Arb Documentation

Release 2.5.0

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CHAPTER ONE

INTRODUCTION

Arb is a C library for arbitrary-precision floating-point ball arithmetic, developed by Fredrik Johansson (fredrik.johansson@gmail.com). It supports real and complex numbers, polynomials, power series, matrices, and evaluation of many transcendental functions. All is done with automatic, rigorous error bounds.

The git repository is https://github.com/fredrik-johansson/arb/

The documentation website is http://fredrikj.net/arb/

CHAPTER

TWO

GENERAL INFORMATION

2.1 Feature overview

Ball arithmetic, also known as mid-rad interval arithmetic, is an extension of floating-point arithmetic in which an error bound is attached to each variable. This allows doing rigorous computations over the real numbers, while avoiding the overhead of traditional (inf-sup) interval arithmetic at high precision, and eliminating much of the need for time-consuming and bug-prone manual error analysis associated with standard floating-point arithmetic. (See for example [Hoe2009].)

Other implementations of ball arithmetic include iRRAM and Mathemagix. In contrast to those systems, Arb is more focused on low-level arithmetic and computation of transcendental functions needed for number theory. Arb also differs in some technical aspects of the implementation.

Arb 2.x contains:

- A module (*arf*) for correctly rounded arbitrary-precision floating-point arithmetic. Arb's floating-point numbers have a few special features, such as arbitrary-size exponents (useful for combinatorics and asymptotics) and dynamic allocation (facilitating implementation of hybrid integer/floating-point and mixed-precision algorithms).
- A module (*mag*) for representing magnitudes (error bounds) more efficiently than with an arbitrary-precision floating-point type.
- A module (*arb*) for real ball arithmetic, where a ball is implemented as an *arf* midpoint and a *mag* radius.
- A module (*acb*) for complex numbers in rectangular form, represented as pairs real balls.
- Functions for fast high-precision evaluation of various mathematical constants and special functions, implemented using ball arithmetic with rigorous error bounds.
- Modules (*arb_poly*, *acb_poly*) for polynomials or power series over the real and complex numbers, implemented using balls as coefficients, with asymptotically fast polynomial multiplication and many other operations.
- Modules (*arb_mat*, *acb_mat*) for matrices over the real and complex numbers, implemented using balls as coefficients. At the moment, only rudimentary linear algebra operations are provided.

Arb 1.x used a different set of numerical base types (*fmpr*, *fmprb* and *fmpcb*). These types had a slightly simpler internal representation, but generally had worse performance. Almost all methods for the Arb 1.x types have now been ported to faster equivalents for the Arb 2.x types. The last version to include both the Arb 1.x and Arb 2.x types and methods was Arb 2.2. As of Arb 2.3, only a small set of *fmpr* and *fmprb* methods are left for fallback and testing purposes.

Planned features include more transcendental functions and more extensive polynomial and matrix functionality, as well as further optimizations.

Arb uses GMP / MPIR and FLINT for the underlying integer arithmetic and other functions. The code conventions borrow from FLINT, and the project might get merged back into FLINT when the code stabilizes in the future. Arb also uses MPFR for testing purposes and for evaluation of some functions.

2.2 Setup

2.2.1 Download

Tarballs of released versions can be downloaded from https://github.com/fredrik-johansson/arb/releases

Alternatively, you can simply install Arb from a git checkout of https://github.com/fredrik-johansson/arb/. The master branch is generally safe to use (the test suite should pass at all times), and recommended for keeping up with the latest changes.

2.2.2 Dependencies

Arb has the following dependencies:

- Either MPIR (http://www.mpir.org) 2.6.0 or later, or GMP (http://www.gmplib.org) 5.1.0 or later. If MPIR is used instead of GMP, it must be compiled with the `--enable-gmpcompat` option.
- MPFR (http://www.mpfr.org) 3.0.0 or later.
- FLINT (http://www.flintlib.org) version 2.4 or later. You may also use a git checkout of https://github.com/fredrik-johansson/flint2

2.2.3 Installation as part of FLINT

With a sufficiently new version of FLINT, Arb can be compiled as a FLINT extension package.

Simply put the Arb source directory somewhere, say `/path/to/arb`. Then go to the FLINT source directory and build FLINT using:

```
./configure --extensions=/path/to/arb <other options>
make
make check       (optional)
make install
```

This is convenient, as Arb does not need to be configured or linked separately. Arb becomes part of the compiled FLINT library, and the Arb header files will be installed along with the other FLINT header files.

2.2.4 Standalone installation

To compile, test and install Arb from source as a standalone library, first install FLINT. Then go to the Arb source directory and run:

```
./configure <options>
make
make check       (optional)
make install
```

If GMP/MPIR, MPFR or FLINT is installed in some other location than the default path `/usr/local`, pass `--with-gmp=...`, `--with-mpfr=...` or `--with-flint=...` with the correct path to configure (type `./configure --help` to show more options).

2.2.5 Running code

Here is an example program to get started using Arb:

```
#include "arb.h"

int main()
{
    arb_t x;
    arb_init(x);
    arb_const_pi(x, 50 * 3.33);
    arb_printn(x, 50); printf("\n");
    printf("Computed with arb-%s\n", arb_version);
    arb_clear(x);
}
```

Compile it with:

```
gcc -larb test.c
```

or (if Arb is built as part of FLINT):

```
gcc -lflint test.c
```

If the Arb/FLINT header and library files are not in a standard location (`/usr/local` on most systems), you may also have to pass options such as:

```
-I/path/to/arb -I/path/to/flint -L/path/to/flint -L/path/to/arb
```

to `gcc`. Finally, to run the program, make sure that the linker can find the FLINT (and Arb) libraries. If they are installed in a nonstandard location, you can for example add this path to the `LD_LIBRARY_PATH` environment variable.

The output of the example program should be something like the following:

```
[3.1415926535897932384626433832795028841971693993751 +/- 6.28e-50]
Computed with arb-2.4.0
```

2.3 Potential issues

2.3.1 Interface changes

Most of the core API should be stable at this point, and significant compatibility-breaking changes will be specified in the release notes.

In general, Arb does not distinguish between "private" and "public" parts of the API. The implementation is meant to be transparent by design. All methods are intended to be fully documented and tested (exceptions to this are mainly due to lack of time on part of the author). The user should use common sense to determine whether a function is concerned with implementation details, making it likely to change as the implementation changes in the future. The interface of `arb_add()` is probably not going to change in the next version, but `_arb_get_mpn_fixed_mod_pi4()` just might.

2.3.2 Correctness

Except where otherwise specified, Arb is designed to produce provably correct error bounds. The code has been written carefully, and the library is extensively tested. However, like any complex mathematical software, Arb is virtually certain to contains bugs, so the usual precautions are advised:

- Perform sanity checks on the output (check known mathematical relations; recompute to another precision and compare)
- Compare against other mathematical software
- Read the source code to verify that it does what it is supposed to do

All bug reports are highly welcome!

2.3.3 Aliasing

As a rule, Arb allows aliasing of operands. For example, in the function call `arb_add(z, x, y, prec)`, which performs $z \leftarrow x + y$, any two (or all three) of the variables *x*, *y* and *z* are allowed to be the same. Exceptions to this rule are documented explicitly.

The general rule that input and output variables can be aliased with each other only applies to variables *of the same type* (ignoring *const* qualifiers on input variables – a special case is that `arb_srcptr` is considered the *const* version of `arb_ptr`). This is a natural extension of the so-called *strict aliasing rule* in C.

For example, in `arb_poly_evaluate()` which evaluates $y = f(x)$ for a polynomial *f*, the output variable *y* is not allowed to be a pointer to one of the coefficients of *f* (but aliasing between *x* and *y* or between *x* and the coefficients of *f* is allowed). This also applies to `_arb_poly_evaluate()`: for the purposes of aliasing, `arb_srcptr` (the type of the coefficient array within *f*) and `arb_t` (the type of *x*) are *not* considered to be the same type, and therefore must not be aliased with each other, even though an `arb_ptr`/`arb_srcptr` variable pointing to a length 1 array would otherwise be interchangeable with an `arb_t`/*const* `arb_t`.

Moreover, in functions that allow aliasing between an input array and an output array, the arrays must either be identical or completely disjoint, never partially overlapping.

There are natural exceptions to these aliasing restrictions, which may used internally without being documented explicitly. However, third party code should avoid relying on such exceptions.

An important caveat applies to **aliasing of input variables**. Identical pointers are understood to give permission for **algebraic simplification**. This assumption is made to improve performance. For example, the call `arb_mul(z, x, x, prec)` sets *z* to a ball enclosing the set

$$\{t^2 \,:\, t \in x\}$$

and not the (generally larger) set

$$\{tu \,:\, t \in x, u \in x\}.$$

If the user knows that two values *x* and *y* both lie in the interval $[-1, 1]$ and wants to compute an enclosure for $f(x, y)$, then it would be a mistake to create an `arb_t` variable *x* enclosing $[-1, 1]$ and reusing the same variable for *y*, calling $f(x, x)$. Instead, the user has to create a distinct variable *y* also enclosing $[-1, 1]$.

Algebraic simplification is not guaranteed to occur. For example, `arb_add(z, x, x, prec)` and `arb_sub(z, x, x, prec)` currently do not implement this optimization. It is better to use `arb_mul_2exp_si(z, x, 1)` and `arb_zero(z)`, respectively.

2.3.4 Integer overflow

Machine-size integers are used for precisions, sizes of integers in bits, lengths of polynomials, and similar quantities that relate to sizes in memory. Very few checks are performed to verify that such quantities do not overflow. Precisions and lengths exceeding a small fraction of *LONG_MAX*, say $2^{24} \sim 10^7$ on 32-bit systems, should be regarded as resulting in undefined behavior. On 64-bit systems this should generally not be an issue, since most calculations will exhaust the available memory (or the user's patience waiting for the computation to complete) long before running into integer overflows. However, the user needs to be wary of unintentionally passing input parameters of order *LONG_MAX* or negative parameters where positive parameters are expected, for example due to a runaway loop that repeatedly increases the precision.

This caveat does not apply to exponents of floating-point numbers, which are represented as arbitrary-precision integers, nor to integers used as numerical scalars (e.g. `arb_mul_si()`). However, it still applies to conversions and operations where the result is requested exactly and sizes become an issue. For example, trying to convert the floating-point number $2^{2^{100}}$ to an integer could result in anything from a silent wrong value to thrashing followed by a crash, and it is the user's responsibility not to attempt such a thing.

2.3.5 Thread safety and caches

Arb should be fully threadsafe, provided that both MPFR and FLINT have been built in threadsafe mode. Use `flint_set_num_threads()` to set the number of threads that Arb is allowed to use internally for single computations (this is currently only exploited by a handful of operations). Please note that thread safety is only tested minimally, and extra caution when developing multithreaded code is therefore recommended.

Arb may cache some data (such as the value of π and Bernoulli numbers) to speed up various computations. In threadsafe mode, caches use thread-local storage. There is currently no way to save memory and avoid recomputation by having several threads share the same cache. Caches can be freed by calling the `flint_cleanup()` function. To avoid memory leaks, the user should call `flint_cleanup()` when exiting a thread. It is also recommended to call `flint_cleanup()` when exiting the main program (this should result in a clean output when running Valgrind, and can help catching memory issues).

There does not seem to be an obvious way to make sure that `flint_cleanup()` is called when exiting a thread using OpenMP. A possible solution to this problem is to use OpenMP sections, or to use C++ and create a thread-local object whose destructor invokes `flint_cleanup()`.

2.3.6 Use of hardware floating-point arithmetic

Arb uses hardware floating-point arithmetic (the `double` type in C) in two different ways.

Firstly, `double` arithmetic as well as transcendental `libm` functions (such as `exp`, `log`) are used to select parameters heuristically in various algorithms. Such heuristic use of approximate arithmetic does not affect correctness: when any error bounds depend on the parameters, the error bounds are evaluated separately using rigorous methods. At worst, flaws in the floating-point arithmetic on a particular machine could cause an algorithm to become inefficient due to inefficient parameters being selected.

Secondly, `double` arithmetic is used internally for some rigorous error bound calculations. To guarantee correctness, we make the following assumptions. With the stated exceptions, these should hold on all commonly used platforms.

- A `double` uses the standard IEEE 754 format (with a 53-bit significand, 11-bit exponent, encoding of infinities and NaNs, etc.)
- We assume that the compiler does not perform "unsafe" floating-point optimizations, such as reordering of operations. Unsafe optimizations are disabled by default in most modern C compilers, including GCC and Clang. The exception appears to be the Intel C++ compiler, which does some unsafe optimizations by default. These must be disabled by the user.
- We do not assume that floating-point operations are correctly rounded (a counterexample is the x87 FPU), or that rounding is done in any particular direction (the rounding mode may have been changed by the user). We assume that any floating-point operation is done with at most 1.1 ulp error.
- We do not assume that underflow or overflow behaves in a particular way (we only use doubles that fit in the regular exponent range, or explicit infinities).

- We do not use transcendental `libm` functions, since these can have errors of several ulps, and there is unfortunately no way to get guaranteed bounds. However, we do use functions such as `ldexp` and `sqrt`, which we assume to be correctly implemented.

2.4 History and changes

For more details, view the commit log in the git repository https://github.com/fredrik-johansson/arb

- 2015-01-28 - version 2.5.0
  - string conversion
    * added arb_set_str
    * added arb_get_str and arb_printn for pretty-printed rigorous decimal output
    * added helper functions for binary to decimal conversion
  - core arithmetic
    * improved speed of division when using GMP instead of MPIR
    * improved complex division with a small denominator
    * removed a little bit of overhead for complex squaring
  - special functions
    * faster code for atan at very high precision, used instead of mpfr_atan
    * optimized elementary functions slightly for small input
    * added modified error functions erfc and erfi
    * added the generalized exponential integral
    * added the upper incomplete gamma function
    * implemented the complete elliptic integral of the first kind
    * implemented the arithmetic-geometric mean of complex numbers
    * optimized arb_digamma for small integers
    * made mag_log_ui, mag_binpow_uiui and mag_polylog_tail proper functions
    * added pow, agm, erf, elliptic_k, elliptic_p as functions of complex power series
    * added incomplete gamma function of complex power series
    * improved code for bounding complex rising factorials (the old code could potentially have given wrong results in degenerate cases)
    * added arb_sqrt1pm1, arb_atanh, arb_asinh, arb_atanh
    * added arb_log1p, acb_log1p, acb_atan
    * added arb_hurwitz_zeta
    * improved parameter selection in the Hurwitz zeta function to try to avoid stalling when given enormous input
    * optimized sqrt and rsqrt of power series when given a binomial as input
    * made arb_bernoulli_ui(2^64-2) not crash
    * fixed rgamma of negative integers returning indeterminate

  - polynomials and matrices
    * added characteristic polynomial computation for real and complex matrices
    * added polynomial set_round methods
    * added is_real methods for more types
    * added more get_unique_fmpz methods
    * added code for generating Swinnerton-Dyer polynomials
    * improved error bounding in det() and exp() of complex matrices to recognize when the result is real-valued
    * changed polynomial divrem to return success/fail instead of aborting on divide by zero
  - miscellaneous
    * added logo to documentation
    * made inlined functions build as part of the library
    * silenced a clang warning
    * made _acb_vec_sort_pretty a library function
- 2014-11-15 - version 2.4.0
  - arithmetic and core functions
    * made evaluation of sin, cos and exp at medium precision faster using the sqrt trick
    * optimized arb_sinh and arb_sinh_cosh
    * optimized complex division with a small denominator
    * optimized cubing of complex numbers
    * added floor and ceil functions for the arf and arb types
    * added acb_poly powering functions
    * added acb_exp_pi_i
    * added functions for evaluation of Chebyshev polynomials
    * fixed arb_div to output nan for input containing nan
  - added a module acb_hypgeom for hypergeometric functions
    * evaluation of the generalized hypergeometric function in convergent cases
    * evaluation of confluent hypergeometric functions using asymptotic expansions
    * the Bessel function of the first kind for complex input
    * the error function for complex input
  - added a module acb_modular for modular forms and elliptic functions
    * support for working with modular transformations
    * mapping a point to the fundamental domain
    * evaluation of Jacobi theta functions and their series expansions
    * the Dedekind eta function
    * the j-invariant and the modular lambda and delta function

    - Eisenstein series
    - the Weierstrass elliptic function and its series expansion
  - miscellaneous
    - fixed mag_print printing a too large exponent
    - fixed printd methods to use a fallback instead of aborting when printing numbers too large for MPFR
    - added version number string (arb_version)
    - various additions to the documentation
- 2014-09-25 - version 2.3.0
  - removed most of the legacy (Arb 1.x) modules
  - updated build scripts, hopefully fixing various issues
  - new implementations of arb_sin, arb_cos, arb_sin_cos, arb_atan, arb_log, arb_exp, arb_expm1, much faster up to a few thousand bits
  - ported the bit-burst code for high-precision exponentials to the arb type
  - speeded up arb_log_ui_from_prev
  - added mag_exp, mag_expm1, mag_exp_tail, mag_pow_fmpz
  - improved various mag functions
  - added arb_get/set_interval_mpfr, arb_get_interval_arf, and improved arb_set_interval_arf
  - improved arf_get_fmpz
  - prettier printing of complex numbers with negative imaginary part
  - changed some frequently-used functions from inline to non-inline to reduce code size
- 2014-08-01 - version 2.2.0
  - added functions for computing polylogarithms and order expansions of polylogarithms, with support for real and complex s, z
  - added a missing cast affecting C++ compatibility
  - generalized powsum functions to allow a geometric factor
  - improved powsum functions slightly when the exponent is an integer
  - faster arb_log_ui_from_prev
  - added mag_sqrt and mag_rsqrt functions
  - fixed various minor bugs and added missing tests and documentation entries
- 2014-06-20 - version 2.1.0
  - ported most of the remaining functions to the new arb/acb types, including:
    - elementary functions (log, atan, etc.)
    - hypergeometric series summation
    - the gamma function
    - the Riemann zeta function and related functions
    - Bernoulli numbers
    - the partition function

        * the calculus modules (rigorous real root isolation, rigorous numerical integration of complex-valued functions)
        * example programs
    - added several missing utility functions to the arf and mag modules

- 2014-05-27 - version 2.0.0
    - new modules mag, arf, arb, arb_poly, arb_mat, acb, acb_poly, acb_mat for higher-performance ball arithmetic
    - poly_roots2 and hilbert_matrix2 example programs
    - vector dot product and norm functions (contributed by Abhinav Baid)
- 2014-05-03 - version 1.1.0
    - faster and more accurate error bounds for polynomial multiplication (error bounds are now always as good as with classical multiplication, and multiplying high-degree polynomials with approximately equal coefficients now has proper quasilinear complexity)
    - faster and much less memory-hungry exponentials at very high precision
    - improved the partition function to support n bigger than a single word, and enabled the possibility to use two threads for the computation
    - fixed a bug in floating-point arithmetic that caused a too small bound for the rounding error to be reported when the result of an inexact operation was rounded up to a power of two (this bug did not affect the correctness of ball arithmetic, because operations on ball midpoints always round down)
    - minor optimizations to floating-point arithmetic
    - improved argument reduction of the digamma function and short series expansions of the rising factorial
    - removed the holonomic module for now, as it did not really do anything very useful
- 2013-12-21 - version 1.0.0
    - new example programs directory
        * poly_roots example program
        * real_roots example program
        * pi_digits example program
        * hilbert_matrix example program
        * keiper_li example program
    - new fmprb_calc module for calculus with real functions
        * bisection-based root isolation
        * asymptotically fast Newton root refinement
    - new fmpcb_calc module for calculus with complex functions
        * numerical integration using Taylor series
    - scalar functions
        * simplified fmprb_const_euler using published error bound
        * added fmprb_inv
        * fmprb_trim, fmpcb_trim

  - added fmpcb_rsqrt (complex reciprocal square root)
  - fixed bug in fmprb_sqrtpos with nonfinite input
  - slightly improved fmprb powering code
  - added various functions for bounding fmprs by powers of two
  - added fmpr_is_int
- polynomials and power series
  - implemented scaling to speed up blockwise multiplication
  - slightly faster basecase power series exponentials
  - improved sin/cos/tan/exp for short power series
  - added complex sqrt_series, rsqrt_series
  - implemented the Riemann-Siegel Z and theta functions for real power series
  - added fmprb_poly_pow_series, fmprb_poly_pow_ui and related methods
  - fmprb/fmpcb_poly_contains_fmpz_poly
  - faster composition by monomials
  - implemented Borel transform and binomial transform for real power series
- matrices
  - implemented matrix exponentials
  - multithreaded fmprb_mat_mul
  - added matrix infinity norm functions
  - added some more matrix-scalar functions
  - added matrix contains and overlaps methods
- zeta function evaluation
  - multithreaded power sum evaluation
  - faster parameter selection when computing many derivatives
  - implemented binary splitting to speed up computing many derivatives
- miscellaneous
  - corrections for C++ compatibility (contributed by Jonathan Bober)
  - several minor bugfixes and test code enhancements

- 2013-08-07 - version 0.7
  - floating-point and ball functions
    - documented, added test code, and fixed bugs for various operations involving a ball containing an infinity or NaN
    - added reciprocal square root functions (fmpr_rsqrt, fmprb_rsqrt) based on mpfr_rec_sqrt
    - faster high-precision division by not computing an explicit remainder
    - slightly faster computation of pi by using new reciprocal square root and division code
    - added an fmpr function for approximate division to speed up certain radius operations

    * added fmpr_set_d for conversion from double
    * allow use of doubles to optionally compute the partition function faster but without an error bound
    * bypass mpfr overflow when computing the exponential function to extremely high precision (approximately 1 billion digits)
    * made fmprb_exp faster for large numbers at extremely high precision by skipping the log(2) removal
    * made fmpcb_lgamma faster at high precision by speeding up the argument reduction branch computation
    * added fmprb_asin, fmprb_acos
    * added various other utility functions to the fmprb module
    * added a function for computing the Glaisher constant
    * optimized evaluation of the Riemann zeta function at high precision
  - polynomials and power series
    * made squaring of polynomials faster than generic multiplication
    * implemented power series reversion (various algorithms) for the fmprb_poly type
    * added many fmprb_poly utility functions (shifting, truncating, setting/getting coefficients, etc.)
    * improved power series division when either operand is short
    * improved power series logarithm when the input is short
    * improved power series exponential to use the basecase algorithm for short input regardless of the output size
    * added power series square root and reciprocal square root
    * added atan, tan, sin, cos, sin_cos, asin, acos fmprb_poly power series functions
    * added Newton iteration macros to simplify various functions
    * added gamma functions of real and complex power series ([fmprb/fmpcb]_poly_[gamma/rgamma/lgamma]_series)
    * added wrappers for computing the Hurwitz zeta function of a power series ([fmprb/fmpcb]_poly_zeta_series)
    * implemented sieving and other optimizations to improve performance for evaluating the zeta function of a short power series
    * improved power series composition when the inner series is linear
    * added many fmpcb_poly versions of nearly all fmprb_poly functions
    * improved speed and stability of series composition/reversion by balancing the power table exponents
  - other
    * added support for freeing all cached data by calling flint_cleanup()
    * introduced fmprb_ptr, fmprb_srcptr, fmpcb_ptr, fmpcb_srcptr typedefs for cleaner function signatures
    * various bug fixes and general cleanup
- 2013-05-31 - version 0.6
  - made fast polynomial multiplication over the reals numerically stable by using a blockwise algorithm

  - disabled default use of the Gauss formula for multiplication of complex polynomials, to improve numerical stability
  - added division and remainder for complex polynomials
  - added fast multipoint evaluation and interpolation for complex polynomials
  - added missing fmprb_poly_sub and fmpcb_poly_sub functions
  - faster exponentials (fmprb_exp and dependent functions) at low precision, using precomputation
  - rewrote fmpr_add and fmpr_sub using mpn level code, improving efficiency at low precision
  - ported the partition function implementation from flint (using ball arithmetic in all steps of the calculation to guarantee correctness)
  - ported algorithm for computing the cosine minimal polynomial from flint (using ball arithmetic to guarantee correctness)
  - support using gmp instead of mpir
  - only use thread-local storage when enabled in flint
  - slightly faster error bounding for the zeta function
  - added some other helper functions

- 2013-03-28 - version 0.5
  - arithmetic and elementary functions
    * added fmpr_get_fmpz, fmpr_get_si
    * fixed accuracy problem with fmprb_div_2expm1
    * special-cased squaring of complex numbers
    * added various fmpcb convenience functions (addmul_ui, etc)
    * optimized fmpr_cmp_2exp_si and fmpr_cmpabs_2exp_si, and added test code for comparison functions
    * added fmprb_atan2, also fixing a bug in fmpcb_arg
    * added fmprb_sin_pi, cos_pi, sin_cos_pi etc.
    * added fmprb_sin_pi_fmpq (etc.) using algebraic methods for fast evaluation of roots of unity
    * faster fmprb_poly_evaluate and evaluate_fmpcb using rectangular splitting
    * added fmprb_poly_evaluate2, evaluate2_fmpcb for simultaneously evaluating the derivative
    * added fmprb_poly root polishing code using near-optimal Newton steps (experimental)
    * added fmpr_root, fmprb_root (currently based on MPFR)
    * added fmpr_min, fmpr_max
    * added fmprb_set_interval_fmpr, fmprb_union
    * added fmpr_bits, fmprb_bits, fmpcb_bits for obtaining the mantissa width
    * added fmprb_hypot
    * added complex square roots
    * improved fmprb_log to slightly improve speed, and properly support huge arguments
    * fixed exp, cosh, sinh to work with huge arguments

        - added fmprb_expm1
        - fixed sin, cos, atan to work with huge arguments
        - improved fmprb_pow and fmpcb_pow, including automatic detection of small integer and half-integer exponents
        - added many more elementary functions: fmprb_tan/cot/tanh/coth, fmpcb_tan/cot, and pi versions
        - added fmprb const_e, const_log2, const_log10, const_catalan
        - fixed ball containment/overlap checking to work operate efficiently and correctly with huge exponents
        - strengthened test code for many core operations
    - special functions
        - reorganized zeta function related code
        - faster evaluation of the Riemann zeta function via sieving
        - documented and improved efficiency of the zeta constant binary splitting code
        - calculate error bound in Borwein's algorithm with fmprs instead of using doubles
        - optimized divisions in zeta evaluation via the Euler product
        - use functional equation for Riemann zeta function of a negative argument
        - compute single Bernoulli numbers using ball arithmetic instead of relying on the floating-point code in flint
        - initial code for evaluating the gamma function using its Taylor series
        - much faster rising factorials at high precision, using difference polynomials
        - much faster gamma function at high precision
        - added complex gamma function, log gamma function, and other versions
        - added fmprb_agm (real arithmetic-geometric mean)
        - added fmprb_gamma_fmpq, supporting rapid computation of gamma(p/q) for q = 1,2,3,4,6
        - added real and complex digamma function
        - fixed unnecessary recomputation of Bernoulli numbers
        - optimized computation of Euler's constant, and added proper error bounds
        - avoid reliance on doubles in the hypergeometric series tail bound
        - cleaned up factorials and binomials, computing factorials via gamma
    - other
        - added an fmpz_extras module to collect various internal fmpz helper functions
        - fixed detection of flint header files
        - fixed various other small bugs
- 2013-01-26 - version 0.4
    - much faster fmpr_mul, fmprb_mul and set_round, resulting in general speed improvements
    - code for computing the complex Hurwitz zeta function with derivatives
    - fixed and documented error bounds for hypergeometric series
    - better algorithm for series evaluation of the gamma function at a rational point

  - much faster generation of Bernoulli numbers
  - complex log, exp, pow, trigonometric functions (currently based on MPFR)
  - complex nth roots via Newton iteration
  - added code for arithmetic on fmpcb_polys
  - code for computing Khinchin's constant
  - code for rising factorials of polynomials or power series
  - faster sin_cos
  - better div_2expm1
  - many other new helper functions
  - improved thread safety
  - more test code for core operations
- 2012-11-07 - version 0.3
  - converted documentation to sphinx
  - new module fmpcb for ball interval arithmetic over the complex numbers
    * conversions, utility functions and arithmetic operations
  - new module fmpcb_mat for matrices over the complex numbers
    * conversions, utility functions and arithmetic operations
    * multiplication, LU decomposition, solving, inverse and determinant
  - new module fmpcb_poly for polynomials over the complex numbers
    * root isolation for complex polynomials
  - new module fmpz_holonomic for functions/sequences defined by linear differential/difference equations with polynomial coefficients
    * functions for creating various special sequences and functions
    * some closure properties for sequences
    * Taylor series expansion for differential equations
    * computing the nth entry of a sequence using binary splitting
    * computing the nth entry mod p using fast multipoint evaluation
  - generic binary splitting code with automatic error bounding is now used for evaluating hypergeometric series
  - matrix powering
  - various other helper functions
- 2012-09-29 - version 0.2
  - code for computing the gamma function (Karatsuba, Stirling's series)
  - rising factorials
  - fast exp_series using Newton iteration
  - improved multiplication of small polynomials by using classical multiplication
  - implemented error propagation for square roots

- polynomial division (Newton-based)
- polynomial evaluation (Horner) and composition (divide-and-conquer)
- product trees, fast multipoint evaluation and interpolation (various algorithms)
- power series composition (Horner, Brent-Kung)
- added the fmprb_mat module for matrices of balls of real numbers
- matrix multiplication
- interval-aware LU decomposition, solving, inverse and determinant
- many helper functions and small bugfixes

- 2012-09-14 - version 0.1
- 2012-08-05 - began simplified rewrite
- 2012-04-05 - experimental ball and polynomial code

2.5 Example programs

The *examples* directory (https://github.com/fredrik-johansson/arb/tree/master/examples) contains several complete C programs, which are documented below. Running:

```
make examples
```

will compile the programs and place the binaries in `build/examples`.

2.5.1 pi.c

This program computes π to an accuracy of roughly n decimal digits by calling the `arb_const_pi()` function with a working precision of roughly $n \log_2(10)$ bits.

Sample output, computing π to one million digits:

```
> build/examples/pi 1000000
computing pi with a precision of 3321933 bits... cpu/wall(s): 0.58 0.586
virt/peak/res/peak(MB): 28.24 36.84 8.86 15.56
[3.14159265358979323846{...999959 digits...}42209010610577945815 +/- 3e-1000000]
```

The program prints an interval guaranteed to contain π, and where all displayed digits are correct up to an error of plus or minus one unit in the last place (see `arb_printn()`). By default, only the first and last few digits are printed. Pass 0 as a second argument to print all digits (or pass m to print $m + 1$ leading and m trailing digits, as above with the default $m = 20$).

2.5.2 hilbert_matrix.c

Given an input integer n, this program accurately computes the determinant of the n by n Hilbert matrix. Hilbert matrices are notoriously ill-conditioned: although the entries are close to unit magnitude, the determinant h_n decreases superexponentially (nearly as $1/4^{n^2}$) as a function of n. This program automatically doubles the working precision until the ball computed for h_n by `arb_mat_det()` does not contain zero.

Sample output:

```
> build/examples/hilbert_matrix 200
prec=20: 0 +/- 5.5777e-330
prec=40: 0 +/- 2.5785e-542
prec=80: 0 +/- 8.1169e-926
prec=160: 0 +/- 2.8538e-1924
prec=320: 0 +/- 6.3868e-4129
prec=640: 0 +/- 1.7529e-8826
prec=1280: 0 +/- 1.8545e-17758
prec=2560: 2.955454297e-23924 +/- 6.4586e-24044
success!
cpu/wall(s): 9.06 9.095
virt/peak/res/peak(MB): 55.52 55.52 35.50 35.50
```

2.5.3 keiper_li.c

Given an input integer *n*, this program rigorously computes numerical values of the Keiper-Li coefficients $\lambda_0, \ldots, \lambda_n$. The Keiper-Li coefficients have the property that $\lambda_n > 0$ for all $n > 0$ if and only if the Riemann hypothesis is true. This program was used for the record computations described in [Joh2013] (the paper describes the algorithm in some more detail).

The program takes the following parameters:

```
keiper_li n [-prec prec] [-threads num_threads] [-out out_file]
```

The program prints the first and last few coefficients. It can optionally write all the computed data to a file. The working precision defaults to a value that should give all the coefficients to a few digits of accuracy, but can optionally be set higher (or lower). On a multicore system, using several threads results in faster execution.

Sample output:

```
> build/examples/keiper_li 1000 -threads 2
zeta: cpu/wall(s): 0.4 0.244
virt/peak/res/peak(MB): 167.98 294.69 5.09 7.43
log: cpu/wall(s): 0.03 0.038
gamma: cpu/wall(s): 0.02 0.016
binomial transform: cpu/wall(s): 0.01 0.018
0: -0.69314718055994530941723212145817656807550013436026 +/- 6.5389e-347
1: 0.023095708966121033814310247906495291621932127152051 +/- 2.0924e-345
2: 0.046172867614023335192864243096033943387066108314123 +/- 1.674e-344
3: 0.0692129735181082679304973488726010689942120263932 +/- 5.0219e-344
4: 0.092197619873060409647627872409439018065541673490213 +/- 2.0089e-343
5: 0.11510854289223549048622128109857276671349132303596 +/- 1.0044e-342
6: 0.13792766871372988290416713700341666356138966078654 +/- 6.0264e-342
7: 0.16063715965299421294040287257385366292282442046163 +/- 2.1092e-341
8: 0.18321945964338257908193931774721859848998098273432 +/- 8.4368e-341
9: 0.20565733870917046170289387421343304741236553410044 +/- 7.5931e-340
10: 0.22793393631931577436930340573684453380748385942738 +/- 7.5931e-339
991: 2.3196617961613367928373899656994682562101430813341 +/- 2.461e-11
992: 2.3203766239254884035349896518332550233162909717288 +/- 9.5363e-11
993: 2.321092061239733282811659116333262802034375592414 +/- 1.8495e-10
994: 2.3218073540188462110258826121503870112747188888893 +/- 3.5907e-10
995: 2.3225217392815185726928702951225314023773358152533 +/- 6.978e-10
996: 2.3232344485814623873333223609413703912358283071281 +/- 1.3574e-09
997: 2.3239447114886014522889542667580382034526509232475 +/- 2.6433e-09
998: 2.3246517591032700808344143240352605148856869322209 +/- 5.1524e-09
999: 2.3253548275861382119812576052060526988544993162101 +/- 1.0053e-08
```

```
1000: 2.3260531616864664574065046940832238158044982041872 +/- 3.927e-08
virt/peak/res/peak(MB): 170.18 294.69 7.51 7.51
```

2.5.4 real_roots.c

This program isolates the roots of a function on the interval (a, b) (where *a* and *b* are input as double-precision literals) using the routines in the *arb_calc* module. The program takes the following arguments:

```
real_roots function a b [-refine d] [-verbose] [-maxdepth n] [-maxeval n] [-maxfound n] [-prec n]
```

The following functions (specified by an integer code) are implemented:

- 0 - $Z(x)$ (Riemann-Siegel Z-function)
- 1 - $\sin(x)$
- 2 - $\sin(x^2)$
- 3 - $\sin(1/x)$

The following options are available:

- `-refine d`: If provided, after isolating the roots, attempt to refine the roots to *d* digits of accuracy using a few bisection steps followed by Newton's method with adaptive precision, and then print them.
- `-verbose`: Print more information.
- `-maxdepth n`: Stop searching after *n* recursive subdivisions.
- `-maxeval n`: Stop searching after approximately *n* function evaluations (the actual number evaluations will be a small multiple of this).
- `-maxfound n`: Stop searching after having found *n* isolated roots.
- `-prec n`: Working precision to use for the root isolation.

With *function* 0, the program isolates roots of the Riemann zeta function on the critical line, and guarantees that no roots are missed (there are more efficient ways to do this, but it is a nice example):

```
> build/examples/real_roots 0 0.0 50.0 -verbose
interval: 25 +/- 25
maxdepth = 30, maxeval = 100000, maxfound = 100000, low_prec = 30
found isolated root in: 14.12353515625 +/- 0.012207
found isolated root in: 21.0205078125 +/- 0.024414
found isolated root in: 25.0244140625 +/- 0.024414
found isolated root in: 30.43212890625 +/- 0.012207
found isolated root in: 32.9345703125 +/- 0.024414
found isolated root in: 37.5732421875 +/- 0.024414
found isolated root in: 40.9423828125 +/- 0.024414
found isolated root in: 43.32275390625 +/- 0.012207
found isolated root in: 48.01025390625 +/- 0.012207
found isolated root in: 49.76806640625 +/- 0.012207
----------------------------------------------------------------
Found roots: 10
Subintervals possibly containing undetected roots: 0
Function evaluations: 3425
cpu/wall(s): 1.22 1.229
virt/peak/res/peak(MB): 20.63 20.66 2.23 2.23
```

Find just one root and refine it to approximately 75 digits:

```
> build/examples/real_roots 0 0.0 50.0 -maxfound 1 -refine 75
interval: 25 +/- 25
maxdepth = 30, maxeval = 100000, maxfound = 1, low_prec = 30
refined root:
14.134725141734693790457251983562470270784257115699243175685567460149963429809 +/- 8.4532e-81

----------------------------------------------------------------
Found roots: 1
Subintervals possibly containing undetected roots: 8
Function evaluations: 992
cpu/wall(s): 0.41 0.415
virt/peak/res/peak(MB): 20.76 20.76 2.23 2.23
```

Find roots of $\sin(x^2)$ on $(0, 100)$. The algorithm cannot isolate the root at $x = 0$ (it is at the endpoint of the interval, and in any case a root of multiplicity higher than one). The failure is reported:

```
> build/examples/real_roots 2 0 100
interval: 50 +/- 50
maxdepth = 30, maxeval = 100000, maxfound = 100000, low_prec = 30
----------------------------------------------------------------
Found roots: 3183
Subintervals possibly containing undetected roots: 1
Function evaluations: 34058
cpu/wall(s): 0.26 0.263
virt/peak/res/peak(MB): 20.73 20.76 1.72 1.72
```

This does not miss any roots:

```
> build/examples/real_roots 2 1 100
interval: 50.5 +/- 49.5
maxdepth = 30, maxeval = 100000, maxfound = 100000, low_prec = 30
----------------------------------------------------------------
Found roots: 3183
Subintervals possibly containing undetected roots: 0
Function evaluations: 34039
cpu/wall(s): 0.26 0.266
virt/peak/res/peak(MB): 20.73 20.76 1.70 1.70
```

Looking for roots of $\sin(1/x)$ on $(0, 1)$, the algorithm finds many roots, but will never find all of them since there are infinitely many:

```
> build/examples/real_roots 3 0.0 1.0
interval: 0.5 +/- 0.5
maxdepth = 30, maxeval = 100000, maxfound = 100000, low_prec = 30
----------------------------------------------------------------
Found roots: 10198
Subintervals possibly containing undetected roots: 24695
Function evaluations: 202587
cpu/wall(s): 1.73 1.731
virt/peak/res/peak(MB): 21.84 22.89 2.76 2.76
```

Remark: the program always computes rigorous containing intervals for the roots, but the accuracy after refinement could be less than *d* digits.

2.5.5 poly_roots.c

This program finds the complex roots of an integer polynomial by calling `acb_poly_find_roots()` with increasing precision until the roots certainly have been isolated. The program takes the following arguments:

```
poly_roots [-refine d] [-print d] <poly>

Isolates all the complex roots of a polynomial with
integer coefficients. For convergence, the input polynomial
is required to be squarefree.

If -refine d is passed, the roots are refined to an absolute
tolerance better than 10^(-d). By default, the roots are only
computed to sufficient accuracy to isolate them.
The refinement is not currently done efficiently.

If -print d is passed, the computed roots are printed to
d decimals. By default, the roots are not printed.

The polynomial can be specified by passing the following as <poly>:

a <n>          Easy polynomial 1 + 2x + ... + (n+1)x^n
t <n>          Chebyshev polynomial T_n
u <n>          Chebyshev polynomial U_n
p <n>          Legendre polynomial P_n
c <n>          Cyclotomic polynomial Phi_n
s <n>          Swinnerton-Dyer polynomial S_n
b <n>          Bernoulli polynomial B_n
w <n>          Wilkinson polynomial W_n
e <n>          Taylor series of exp(x) truncated to degree n
m <n> <m>      The Mignotte-like polynomial x^n + (100x+1)^m, n > m
c0 c1 ... cn   c0 + c1 x + ... + cn x^n where all c:s are specified integers
```

This finds the roots of the Wilkinson polynomial with roots at the positive integers 1, 2, ..., 100:

```
> build/examples/poly_roots -print 15 w 100
prec=53: 0 isolated roots | cpu/wall(s): 0.42 0.426
prec=106: 0 isolated roots | cpu/wall(s): 1.37 1.368
prec=212: 0 isolated roots | cpu/wall(s): 1.48 1.485
prec=424: 100 isolated roots | cpu/wall(s): 0.61 0.611
done!
(1 + 1.7285178043492e-125j)  +/-  (7.2e-122, 7.2e-122j)
(2 + 5.1605530263601e-122j)  +/-  (3.77e-118, 3.77e-118j)
(3 + -2.58115555871665e-118j)  +/-  (5.72e-115, 5.72e-115j)
(4 + 1.02141628524271e-115j)  +/-  (4.38e-112, 4.38e-112j)
(5 + 1.61326834094948e-113j)  +/-  (2.6e-109, 2.6e-109j)
    ...
(95 + 4.15294196875447e-62j)  +/-  (6.66e-59, 6.66e-59j)
(96 + 3.54502401922667e-64j)  +/-  (7.37e-60, 7.37e-60j)
(97 + -1.67755595325625e-65j)  +/-  (6.4e-61, 6.4e-61j)
(98 + 2.04638822325299e-65j)  +/-  (4e-62, 4e-62j)
(99 + -2.73425468028238e-66j)  +/-  (1.71e-63, 1.71e-63j)
(100 + -1.00950111302288e-68j)  +/-  (3.24e-65, 3.24e-65j)
cpu/wall(s): 3.88 3.893
```

This finds the roots of a Bernoulli polynomial which has both real and complex roots. Note that the program does not attempt to determine that the imaginary parts of the real roots really are zero (this could be done by verifying sign changes):

```
> build/examples/poly_roots -refine 100 -print 20 b 16
prec=53: 16 isolated roots | cpu/wall(s): 0 0.007
prec=106: 16 isolated roots | cpu/wall(s): 0 0.004
prec=212: 16 isolated roots | cpu/wall(s): 0 0.004
prec=424: 16 isolated roots | cpu/wall(s): 0 0.004
```

```
done!
(-0.94308706466055783383 + -5.512272663168484603e-128j)  +/-  (2.2e-125, 2.2e-125j)
(-0.75534059252067985752 + 1.937401283040249068e-128j)  +/-  (1.09e-125, 1.09e-125j)
(-0.24999757119077421009 + -4.5347924422246038692e-130j)  +/-  (3.6e-127, 3.6e-127j)
(0.24999757152512726002 + 4.2191300761823281708e-129j)  +/-  (4.98e-127, 4.98e-127j)
(0.75000242847487273998 + 9.0360649917413170142e-128j)  +/-  (8.88e-126, 8.88e-126j)
(1.2499975711907742101 + 7.8804123808107088267e-127j)  +/-  (2.66e-124, 2.66e-124j)
(1.7553405925206798575 + 5.432465269253967768e-126j)  +/-  (6.23e-123, 6.23e-123j)
(1.9430870646605578338 + 3.3035377342500953239e-125j)  +/-  (7.05e-123, 7.05e-123j)
(-0.99509334829256233279 + 0.44547958157103608805j)  +/-  (5.5e-125, 5.5e-125j)
(-0.99509334829256233279 + -0.44547958157103608805j)  +/-  (5.46e-125, 5.46e-125j)
(1.9950933482925623328 + 0.44547958157103608805j)  +/-  (1.44e-122, 1.44e-122j)
(1.9950933482925623328 + -0.44547958157103608805j)  +/-  (1.43e-122, 1.43e-122j)
(-0.92177327714429290564 + -1.0954360955079385542j)  +/-  (9.31e-125, 9.31e-125j)
(-0.92177327714429290564 + 1.0954360955079385542j)  +/-  (1.02e-124, 1.02e-124j)
(1.9217732771442929056 + 1.0954360955079385542j)  +/-  (9.15e-123, 9.15e-123j)
(1.9217732771442929056 + -1.0954360955079385542j)  +/-  (8.12e-123, 8.12e-123j)
cpu/wall(s): 0.02 0.02
```

CHAPTER

THREE

MODULE DOCUMENTATION (ARB 2.X TYPES)

3.1 mag.h – fixed-precision unsigned floating-point numbers for bounds

The `mag_t` type is an unsigned floating-point type with a fixed-precision mantissa (30 bits) and an arbitrary-precision exponent (represented as an `fmpz_t`), suited for representing and rigorously manipulating magnitude bounds efficiently. Operations always produce a strict upper or lower bound, but for performance reasons, no attempt is made to compute the best possible bound (in general, a result may a few ulps larger/smaller than the optimal value). The special values zero and positive infinity are supported (but not NaN). Applications requiring more flexibility (such as correct rounding, or higher precision) should use the `arf_t` type instead.

3.1.1 Types, macros and constants

mag_struct
A `mag_struct` holds a mantissa and an exponent. Special values are encoded by the mantissa being set to zero.

mag_t
A `mag_t` is defined as an array of length one of type `mag_struct`, permitting a `mag_t` to be passed by reference.

3.1.2 Memory management

void **mag_init** (mag_t *x*)
Initializes the variable *x* for use. Its value is set to zero.

void **mag_clear** (mag_t *x*)
Clears the variable *x*, freeing or recycling its allocated memory.

void **mag_init_set** (mag_t *x*, const mag_t *y*)
Initializes *x* and sets it to the value of *y*.

void **mag_swap** (mag_t *x*, mag_t *y*)
Swaps *x* and *y* efficiently.

void **mag_set** (mag_t *x*, const mag_t *y*)
Sets *x* to the value of *y*.

mag_ptr **_mag_vec_init** (long *n*)
Allocates a vector of length *n*. All entries are set to zero.

void **_mag_vec_clear** (mag_ptr *v*, long *n*)
Clears a vector of length *n*.

3.1.3 Special values

void **mag_zero** (mag_t *x*)
Sets *x* to zero.

void **mag_one** (mag_t *x*)
Sets *x* to one.

void **mag_inf** (mag_t *x*)
Sets *x* to positive infinity.

int **mag_is_special** (const mag_t *x*)
Returns nonzero iff *x* is zero or positive infinity.

int **mag_is_zero** (const mag_t *x*)
Returns nonzero iff *x* is zero.

int **mag_is_inf** (const mag_t *x*)
Returns nonzero iff *x* is positive infinity.

int **mag_is_finite** (const mag_t *x*)
Returns nonzero iff *x* is not positive infinity (since there is no NaN value, this function is exactly the negation of `mag_is_inf()`).

3.1.4 Comparisons

int **mag_equal** (const mag_t *x*, const mag_t *y*)
Returns nonzero iff *x* and *y* have the same value.

int **mag_cmp** (const mag_t *x*, const mag_t *y*)
Returns negative, zero, or positive, depending on whether *x* is smaller, equal, or larger than *y*.

int **mag_cmp_2exp_si** (const mag_t *x*, long *y*)
Returns negative, zero, or positive, depending on whether *x* is smaller, equal, or larger than 2^y.

void **mag_min** (mag_t *z*, const mag_t *x*, const mag_t *y*)

void **mag_max** (mag_t *z*, const mag_t *x*, const mag_t *y*)
Sets *z* respectively to the smaller or the larger of *x* and *y*.

3.1.5 Input and output

void **mag_print** (const mag_t *x*)
Prints *x* to standard output.

3.1.6 Random generation

void **mag_randtest** (mag_t *x*, flint_rand_t *state*, long *expbits*)
Sets *x* to a random finite value, with an exponent up to *expbits* bits large.

void **mag_randtest_special** (mag_t *x*, flint_rand_t *state*, long *expbits*)
Like `mag_randtest()`, but also sometimes sets *x* to infinity.

3.1.7 Conversions

void **mag_set_d** (mag_t *y*, double *x*)

void **mag_set_fmpr** (mag_t *y*, const fmpr_t *x*)

void **mag_set_ui** (mag_t *y*, ulong *x*)

void **mag_set_fmpz** (mag_t *y*, const fmpz_t *x*)
Sets *y* to an upper bound for $|x|$.

void **mag_set_d_2exp_fmpz** (mag_t *z*, double *x*, const fmpz_t *y*)

void **mag_set_fmpz_2exp_fmpz** (mag_t *z*, const fmpz_t *x*, const fmpz_t *y*)

void **mag_set_ui_2exp_si** (mag_t *z*, ulong *x*, long *y*)
Sets *z* to an upper bound for $|x| \times 2^y$.

void **mag_get_fmpr** (fmpr_t *y*, const mag_t *x*)
Sets *y* exactly to *x*.

void **mag_get_fmpq** (fmpq_t *y*, const mag_t *x*)
Sets *y* exactly to x. Assumes that no overflow occurs.

void **mag_set_ui_lower** (mag_t *z*, ulong *x*)

void **mag_set_fmpz_lower** (mag_t *z*, const fmpz_t *x*)
Sets *y* to a lower bound for $|x|$.

void **mag_set_fmpz_2exp_fmpz_lower** (mag_t *z*, const fmpz_t *x*, const fmpz_t *y*)
Sets *z* to a lower bound for $|x| \times 2^y$.

3.1.8 Arithmetic

void **mag_mul_2exp_si** (mag_t *z*, const mag_t *x*, long *y*)

void **mag_mul_2exp_fmpz** (mag_t *z*, const mag_t *x*, const fmpz_t *y*)
Sets *z* to $x \times 2^y$. This operation is exact.

void **mag_mul** (mag_t *z*, const mag_t *x*, const mag_t *y*)

void **mag_mul_ui** (mag_t *z*, const mag_t *x*, ulong *y*)

void **mag_mul_fmpz** (mag_t *z*, const mag_t *x*, const fmpz_t *y*)
Sets *z* to an upper bound for xy.

void **mag_add** (mag_t *z*, const mag_t *x*, const mag_t *y*)
Sets *z* to an upper bound for $x + y$.

void **mag_addmul** (mag_t *z*, const mag_t *x*, const mag_t *y*)
Sets *z* to an upper bound for $z + xy$.

void **mag_add_2exp_fmpz** (mag_t *z*, const mag_t *x*, const fmpz_t *e*)
Sets *z* to an upper bound for $x + 2^e$.

void **mag_div** (mag_t *z*, const mag_t *x*, const mag_t *y*)

void **mag_div_ui** (mag_t *z*, const mag_t *x*, ulong *y*)

void **mag_div_fmpz** (mag_t *z*, const mag_t *x*, const fmpz_t *y*)
Sets *z* to an upper bound for x/y.

void **mag_mul_lower** (mag_t *z*, const mag_t *x*, const mag_t *y*) void **mag_mul_ui_lower** (mag_t *z*, const mag_t *x*, ulong *y*)

void **mag_mul_fmpz_lower** (mag_t *z*, const mag_t *x*, const fmpz_t *y*)
Sets *z* to a lower bound for xy.

void **mag_add_lower** (mag_t *z*, const mag_t *x*, const mag_t *y*)
Sets *z* to a lower bound for $x + y$.

void **mag_sub_lower** (mag_t *z*, const mag_t *x*, const mag_t *y*)
Sets *z* to a lower bound for $\max(x - y, 0)$.

3.1.9 Fast, unsafe arithmetic

The following methods assume that all inputs are finite and that all exponents (in all inputs as well as the final result) fit as *fmpz* inline values. They also assume that the output variables do not have promoted exponents, as they will be overwritten directly (thus leaking memory).

void **mag_fast_init_set** (mag_t *x*, const mag_t *y*)
Initialises *x* and sets it to the value of *y*.

void **mag_fast_zero** (mag_t *x*)
Sets *x* to zero.

int **mag_fast_is_zero** (const mag_t *x*)
Returns nonzero iff *x* to zero.

void **mag_fast_mul** (mag_t *z*, const mag_t *x*, const mag_t *y*)
Sets *z* to an upper bound for xy.

void **mag_fast_addmul** (mag_t *z*, const mag_t *x*, const mag_t *y*)
Sets *z* to an upper bound for $z + xy$.

void **mag_fast_add_2exp_si** (mag_t *z*, const mag_t *x*, long *e*)
Sets *z* to an upper bound for $x + 2^e$.

void **mag_fast_mul_2exp_si** (mag_t *z*, const mag_t *x*, long *e*)
Sets *z* to an upper bound for $x2^e$.

3.1.10 Powers and logarithms

void **mag_pow_ui** (mag_t *z*, const mag_t *x*, ulong *e*)

void **mag_pow_fmpz** (mag_t *z*, const mag_t *x*, const fmpz_t *e*)
Sets *z* to an upper bound for x^e. Requires $e \geq 0$.

void **mag_pow_ui_lower** (mag_t *z*, const mag_t *x*, ulong *e*)
Sets *z* to a lower bound for x^e.

void **mag_sqrt** (mag_t *z*, const mag_t *x*)
Sets *z* to an upper bound for $\sqrt{x}$.

void **mag_rsqrt** (mag_t *z*, const mag_t *x*)
Sets *z* to an upper bound for $1/\sqrt{x}$.

void **mag_hypot** (mag_t *z*, const mag_t *x*, const mag_t *y*)
Sets *z* to an upper bound for $\sqrt{x^2 + y^2}$.

void **mag_log1p** (mag_t *z*, const mag_t *x*)
Sets *z* to an upper bound for $\log(1 + x)$. The bound is computed accurately for small *x*.

void **mag_log_ui** (mag_t *z*, ulong *n*)
Sets *z* to an upper bound for $\log(n)$.

void **mag_exp** (mag_t *z*, const mag_t *x*)
Sets *z* to an upper bound for $\exp(x)$.

void **mag_expm1** (mag_t *z*, const mag_t *x*)
Sets *z* to an upper bound for $\exp(x) - 1$. The bound is computed accurately for small *x*.

void **mag_exp_tail** (mag_t *z*, const mag_t *x*, ulong *N*)
Sets *z* to an upper bound for $\sum_{k=N}^{\infty} x^k/k!$.

void **mag_binpow_uiui** (mag_t *z*, ulong *m*, ulong *n*)
Sets *z* to an upper bound for $(1 + 1/m)^n$.

3.1.11 Special functions

void **mag_fac_ui** (mag_t *z*, ulong *n*)
Sets *z* to an upper bound for $n!$.

void **mag_rfac_ui** (mag_t *z*, ulong *n*)
Sets *z* to an upper bound for $1/n!$.

void **mag_bernoulli_div_fac_ui** (mag_t *z*, ulong *n*)
Sets *z* to an upper bound for $|B_n|/n!$ where B_n denotes a Bernoulli number.

void **mag_polylog_tail** (mag_t *u*, const mag_t *z*, long *s*, ulong *d*, ulong *N*)
Sets *u* to an upper bound for

$$\sum_{k=N}^{\infty} \frac{z^k \log^d(k)}{k^s}.$$

Note: in applications where s in this formula may be real or complex, the user can simply substitute any convenient integer s' such that $s' \leq \operatorname{Re}(s)$.

Denote the terms by $T(k)$. We pick a nonincreasing function $U(k)$ such that

$$\frac{T(k+1)}{T(k)} = z \left(\frac{k}{k+1}\right)^s \left(\frac{\log(k+1)}{\log(k)}\right)^d \leq U(k).$$

Then, as soon as $U(N) < 1$,

$$\sum_{k=N}^{\infty} T(k) \leq T(N) \sum_{k=0}^{\infty} U(N)^k = \frac{T(N)}{1 - U(N)}.$$

In particular, we take

$$U(k) = z\; B(k, \max(0, -s))\; B(k \log(k), d)$$

where $B(m, n) = (1 + 1/m)^n$. This follows from the bounds

$$\left(\frac{k}{k+1}\right)^s \leq \begin{cases} 1 & \text{if } s \geq 0 \\ (1 + 1/k)^{-s} & \text{if } s < 0. \end{cases}$$

and

$$\left(\frac{\log(k+1)}{\log(k)}\right)^d \leq \left(1 + \frac{1}{k \log(k)}\right)^d.$$

3.2 arf.h – arbitrary-precision floating-point numbers

A variable of type arf_t holds an arbitrary-precision binary floating-point number, i.e. a rational number of the form $x \times 2^y$ where $x, y \in \mathbb{Z}$ and x is odd; or one of the special values zero, plus infinity, minus infinity, or NaN (not-a-number).

The *exponent* of a finite and nonzero floating-point number can be defined in different ways: for example, as the component *y* above, or as the unique integer *e* such that $x \times 2^y = m \times 2^e$ where $1/2 \le |m| < 1$. The internal representation of an arf_t stores the exponent in the latter format.

The conventions for special values largely follow those of the IEEE floating-point standard. At the moment, there is no support for negative zero, unsigned infinity, or a NaN with a payload, though some these might be added in the future.

Except where otherwise noted, the output of an operation is the floating-point number obtained by taking the inputs as exact numbers, in principle carrying out the operation exactly, and rounding the resulting real number to the nearest representable floating-point number whose mantissa has at most the specified number of bits, in the specified direction of rounding. Some operations are always or optionally done exactly.

The arf_t type is almost identical semantically to the legacy fmpr_t type, but uses a more efficient internal representation. The most significant differences that the user has to be aware of are:

- The mantissa is no longer represented as a FLINT fmpz, and the internal exponent points to the top of the binary expansion of the mantissa instead of of the bottom. Code designed to manipulate components of an fmpr_t directly can be ported to the arf_t type by making use of arf_get_fmpz_2exp() and arf_set_fmpz_2exp().
- Some arf_t functions return an int indicating whether a result is inexact, whereas the corresponding fmpr_t functions return a long encoding the relative exponent of the error.

3.2.1 Types, macros and constants

arf_struct

arf_t

An arf_struct contains four words: an fmpz exponent (*exp*), a *size* field tracking the number of limbs used (one bit of this field is also used for the sign of the number), and two more words. The last two words hold the value directly if there are at most two limbs, and otherwise contain one *alloc* field (tracking the total number of allocated limbs, not all of which might be used) and a pointer to the actual limbs. Thus, up to 128 bits on a 64-bit machine and 64 bits on a 32-bit machine, no space outside of the arf_struct is used.

An arf_t is defined as an array of length one of type arf_struct, permitting an arf_t to be passed by reference.

arf_rnd_t

Specifies the rounding mode for the result of an approximate operation.

ARF_RND_DOWN

Specifies that the result of an operation should be rounded to the nearest representable number in the direction towards zero.

ARF_RND_UP

Specifies that the result of an operation should be rounded to the nearest representable number in the direction away from zero.

ARF_RND_FLOOR

Specifies that the result of an operation should be rounded to the nearest representable number in the direction towards minus infinity.

ARF_RND_CEIL
Specifies that the result of an operation should be rounded to the nearest representable number in the direction towards plus infinity.

ARF_RND_NEAR
Specifies that the result of an operation should be rounded to the nearest representable number, rounding to an odd mantissa if there is a tie between two values. *Warning*: this rounding mode is currently not implemented (except for a few conversions functions where this stated explicitly).

ARF_PREC_EXACT
If passed as the precision parameter to a function, indicates that no rounding is to be performed. This must only be used when it is known that the result of the operation can be represented exactly and fits in memory (the typical use case is working with small integer values). Note that, for example, adding two numbers whose exponents are far apart can easily produce an exact result that is far too large to store in memory.

3.2.2 Memory management

void **arf_init** (arf_t *x*)
Initializes the variable *x* for use. Its value is set to zero.

void **arf_clear** (arf_t *x*)
Clears the variable *x*, freeing or recycling its allocated memory.

3.2.3 Special values

void **arf_zero** (arf_t *x*)

void **arf_one** (arf_t *x*)

void **arf_pos_inf** (arf_t *x*)

void **arf_neg_inf** (arf_t *x*)

void **arf_nan** (arf_t *x*)
Sets *x* respectively to 0, 1, $+\infty$, $-\infty$, NaN.

int **arf_is_zero** (const arf_t *x*)

int **arf_is_one** (const arf_t *x*)

int **arf_is_pos_inf** (const arf_t *x*)

int **arf_is_neg_inf** (const arf_t *x*)

int **arf_is_nan** (const arf_t *x*)
Returns nonzero iff *x* respectively equals 0, 1, $+\infty$, $-\infty$, NaN.

int **arf_is_inf** (const arf_t *x*)
Returns nonzero iff *x* equals either $+\infty$ or $-\infty$.

int **arf_is_normal** (const arf_t *x*)
Returns nonzero iff *x* is a finite, nonzero floating-point value, i.e. not one of the special values 0, $+\infty$, $-\infty$, NaN.

int **arf_is_special** (const arf_t *x*)
Returns nonzero iff *x* is one of the special values 0, $+\infty$, $-\infty$, NaN, i.e. not a finite, nonzero floating-point value.

int **arf_is_finite** (arf_t *x*)
Returns nonzero iff *x* is a finite floating-point value, i.e. not one of the values $+\infty$, $-\infty$, NaN. (Note that this is not equivalent to the negation of `arf_is_inf()`.)

3.2.4 Assignment, rounding and conversions

void **arf_set** (arf_t *y*, const arf_t *x*)

void **arf_set_mpz** (arf_t *y*, const mpz_t *x*)

void **arf_set_fmpz** (arf_t *y*, const fmpz_t *x*)

void **arf_set_ui** (arf_t *y*, ulong *x*)

void **arf_set_si** (arf_t *y*, long *x*)

void **arf_set_mpfr** (arf_t *y*, const mpfr_t *x*)

void **arf_set_fmpr** (arf_t *y*, const fmpr_t *x*)

void **arf_set_d** (arf_t *y*, double *x*)
Sets *y* exactly to *x*.

void **arf_swap** (arf_t *y*, arf_t *x*)
Swaps *y* and *x* efficiently.

void **arf_init_set_ui** (arf_t *y*, ulong *x*)

void **arf_init_set_si** (arf_t *y*, long *x*)
Initialises *y* and sets it to *x* in a single operation.

int **arf_set_round** (arf_t *y*, const arf_t *x*, long *prec*, arf_rnd_t *rnd*)

int **arf_set_round_si** (arf_t *x*, long *v*, long *prec*, arf_rnd_t *rnd*)

int **arf_set_round_ui** (arf_t *x*, ulong *v*, long *prec*, arf_rnd_t *rnd*)

int **arf_set_round_mpz** (arf_t *y*, const mpz_t *x*, long *prec*, arf_rnd_t *rnd*)

int **arf_set_round_fmpz** (arf_t *y*, const fmpz_t *x*, long *prec*, arf_rnd_t *rnd*)
Sets *y* to *x*, rounded to *prec* bits in the direction specified by *rnd*.

void **arf_set_si_2exp_si** (arf_t *y*, long *m*, long *e*)

void **arf_set_ui_2exp_si** (arf_t *y*, ulong *m*, long *e*)

void **arf_set_fmpz_2exp** (arf_t *y*, const fmpz_t *m*, const fmpz_t *e*)
Sets *y* to $m \times 2^e$.

int **arf_set_round_fmpz_2exp** (arf_t *y*, const fmpz_t *x*, const fmpz_t *e*, long *prec*, arf_rnd_t *rnd*)
Sets *y* to $x \times 2^e$, rounded to *prec* bits in the direction specified by *rnd*.

void **arf_get_fmpz_2exp** (fmpz_t *m*, fmpz_t *e*, const arf_t *x*)
Sets *m* and *e* to the unique integers such that $x = m \times 2^e$ and *m* is odd, provided that *x* is a nonzero finite fraction. If *x* is zero, both *m* and *e* are set to zero. If *x* is infinite or NaN, the result is undefined.

double **arf_get_d** (const arf_t *x*, arf_rnd_t *rnd*)
Returns *x* rounded to a double in the direction specified by *rnd*.

void **arf_get_fmpr** (fmpr_t *y*, const arf_t *x*)
Sets *y* exactly to *x*.

int **arf_get_mpfr** (mpfr_t *y*, const arf_t *x*, mpfr_rnd_t *rnd*)
Sets the MPFR variable *y* to the value of *x*. If the precision of *x* is too small to allow *y* to be represented exactly, it is rounded in the specified MPFR rounding mode. The return value (-1, 0 or 1) indicates the direction of rounding, following the convention of the MPFR library.

void **arf_get_fmpz** (fmpz_t *z*, const arf_t *x*, arf_rnd_t *rnd*)
Sets *z* to *x* rounded to the nearest integer in the direction specified by *rnd*. If rnd is *ARF_RND_NEAR*, rounds to the nearest even integer in case of a tie. Aborts if *x* is infinite, NaN or if the exponent is unreasonably large.

long **arf_get_si** (const arf_t *x*, arf_rnd_t *rnd*)
Returns *x* rounded to the nearest integer in the direction specified by *rnd*. If *rnd* is *ARF_RND_NEAR*, rounds to the nearest even integer in case of a tie. Aborts if *x* is infinite, NaN, or the value is too large to fit in a long.

int **arf_get_fmpz_fixed_fmpz** (fmpz_t *y*, const arf_t *x*, const fmpz_t *e*)

int **arf_get_fmpz_fixed_si** (fmpz_t *y*, const arf_t *x*, long *e*)
Converts *x* to a mantissa with predetermined exponent, i.e. computes an integer *y* such that $y \times 2^e \approx x$, truncating if necessary. Returns 0 if exact and 1 if truncation occurred.

void **arf_floor** (arf_t *y*, const arf_t *x*)

void **arf_ceil** (arf_t *y*, const arf_t *x*)
Sets *y* to $\lfloor x \rfloor$ and $\lceil x \rceil$ respectively. The result is always represented exactly, requiring no more bits to store than the input. To round the result to a floating-point number with a lower precision, call `arf_set_round()` afterwards.

3.2.5 Comparisons and bounds

int **arf_equal** (const arf_t *x*, const arf_t *y*)

int **arf_equal_si** (const arf_t *x*, long *y*)
Returns nonzero iff *x* and *y* are exactly equal. This function does not treat NaN specially, i.e. NaN compares as equal to itself.

int **arf_cmp** (const arf_t *x*, const arf_t *y*)
Returns negative, zero, or positive, depending on whether *x* is respectively smaller, equal, or greater compared to *y*. Comparison with NaN is undefined.

int **arf_cmpabs** (const arf_t *x*, const arf_t *y*)

int **arf_cmpabs_ui** (const arf_t *x*, ulong *y*)

int **arf_cmpabs_mag** (const arf_t *x*, const mag_t *y*)
Compares the absolute values of *x* and *y*.

int **arf_cmp_2exp_si** (const arf_t *x*, long *e*)

int **arf_cmpabs_2exp_si** (const arf_t *x*, long *e*)
Compares *x* (respectively its absolute value) with 2^e.

int **arf_sgn** (const arf_t *x*)
Returns -1, 0 or $+1$ according to the sign of *x*. The sign of NaN is undefined.

void **arf_min** (arf_t *z*, const arf_t *a*, const arf_t *b*)

void **arf_max** (arf_t *z*, const arf_t *a*, const arf_t *b*)
Sets *z* respectively to the minimum and the maximum of *a* and *b*.

long **arf_bits** (const arf_t *x*)
Returns the number of bits needed to represent the absolute value of the mantissa of *x*, i.e. the minimum precision sufficient to represent *x* exactly. Returns 0 if *x* is a special value.

int **arf_is_int** (const arf_t *x*)
Returns nonzero iff *x* is integer-valued.

int **arf_is_int_2exp_si** (const arf_t *x*, long *e*)
Returns nonzero iff *x* equals $n2^e$ for some integer *n*.

void **arf_abs_bound_lt_2exp_fmpz** (fmpz_t *b*, const arf_t *x*)
Sets *b* to the smallest integer such that $|x| < 2^b$. If *x* is zero, infinity or NaN, the result is undefined.

void **arf_abs_bound_le_2exp_fmpz** (fmpz_t *b*, const arf_t *x*)
Sets *b* to the smallest integer such that $|x| \leq 2^b$. If *x* is zero, infinity or NaN, the result is undefined.

long **arf_abs_bound_lt_2exp_si** (const arf_t *x*)
Returns the smallest integer *b* such that $|x| < 2^b$, clamping the result to lie between *-ARF_PREC_EXACT* and *ARF_PREC_EXACT* inclusive. If *x* is zero, *-ARF_PREC_EXACT* is returned, and if *x* is infinity or NaN, *ARF_PREC_EXACT* is returned.

3.2.6 Magnitude functions

void **arf_get_mag** (mag_t *y*, const arf_t *x*)
Sets *y* to an upper bound for the absolute value of *x*.

void **arf_get_mag_lower** (mag_t *y*, const arf_t *x*)
Sets *y* to a lower bound for the absolute value of *x*.

void **arf_set_mag** (arf_t *y*, const mag_t *x*)
Sets *y* to *x*.

void **mag_init_set_arf** (mag_t *y*, const arf_t *x*)
Initializes *y* and sets it to an upper bound for *x*.

void **mag_fast_init_set_arf** (mag_t *y*, const arf_t *x*)
Initializes *y* and sets it to an upper bound for *x*. Assumes that the exponent of *y* is small.

void **arf_mag_set_ulp** (mag_t *z*, const arf_t *y*, long *prec*)
Sets *z* to the magnitude of the unit in the last place (ulp) of *y* at precision *prec*.

void **arf_mag_add_ulp** (mag_t *z*, const mag_t *x*, const arf_t *y*, long *prec*)
Sets *z* to an upper bound for the sum of *x* and the magnitude of the unit in the last place (ulp) of *y* at precision *prec*.

void **arf_mag_fast_add_ulp** (mag_t *z*, const mag_t *x*, const arf_t *y*, long *prec*)
Sets *z* to an upper bound for the sum of *x* and the magnitude of the unit in the last place (ulp) of *y* at precision *prec*. Assumes that all exponents are small.

3.2.7 Shallow assignment

void **arf_init_set_shallow** (arf_t *z*, const arf_t *x*)

void **arf_init_set_mag_shallow** (arf_t *z*, const mag_t *x*)
Initializes *z* to a shallow copy of *x*. A shallow copy just involves copying struct data (no heap allocation is performed).

The target variable *z* may not be cleared or modified in any way (it can only be used as constant input to functions), and may not be used after *x* has been cleared. Moreover, after *x* has been assigned shallowly to *z*, no modification of *x* is permitted as long as *z* is in use.

void **arf_init_neg_shallow** (arf_t *z*, const arf_t *x*)

void **arf_init_neg_mag_shallow** (arf_t *z*, const mag_t *x*)
Initializes *z* shallowly to the negation of *x*.

3.2.8 Random number generation

void **arf_randtest** (arf_t *x*, flint_rand_t *state*, long *bits*, long *mag_bits*)
Generates a finite random number whose mantissa has precision at most *bits* and whose exponent has at most *mag_bits* bits. The values are distributed non-uniformly: special bit patterns are generated with high probability in order to allow the test code to exercise corner cases.

void **arf_randtest_not_zero** (arf_t *x*, flint_rand_t *state*, long *bits*, long *mag_bits*)
Identical to `arf_randtest()`, except that zero is never produced as an output.

void **arf_randtest_special** (arf_t *x*, flint_rand_t *state*, long *bits*, long *mag_bits*)
Indentical to `arf_randtest()`, except that the output occasionally is set to an infinity or NaN.

3.2.9 Input and output

void **arf_debug** (const arf_t *x*)
Prints information about the internal representation of *x*.

void **arf_print** (const arf_t *x*)
Prints *x* as an integer mantissa and exponent.

void **arf_printd** (const arf_t *y*, long *d*)
Prints *x* as a decimal floating-point number, rounding to *d* digits. This function is currently implemented using MPFR, and does not support large exponents.

3.2.10 Addition and multiplication

void **arf_abs** (arf_t *y*, const arf_t *x*)
Sets *y* to the absolute value of *x*.

void **arf_neg** (arf_t *y*, const arf_t *x*)
Sets $y = -x$ exactly.

int **arf_neg_round** (arf_t *y*, const arf_t *x*, long *prec*, arf_rnd_t *rnd*)
Sets $y = -x$, rounded to *prec* bits in the direction specified by *rnd*, returning nonzero iff the operation is inexact.

void **arf_mul_2exp_si** (arf_t *y*, const arf_t *x*, long *e*)

void **arf_mul_2exp_fmpz** (arf_t *y*, const arf_t *x*, const fmpz_t *e*)
Sets $y = x2^e$ exactly.

int **arf_mul** (arf_t *z*, const arf_t *x*, const arf_t *y*, long *prec*, arf_rnd_t *rnd*)

int **arf_mul_ui** (arf_t *z*, const arf_t *x*, ulong *y*, long *prec*, arf_rnd_t *rnd*)

int **arf_mul_si** (arf_t *z*, const arf_t *x*, long *y*, long *prec*, arf_rnd_t *rnd*)

int **arf_mul_mpz** (arf_t *z*, const arf_t *x*, const mpz_t *y*, long *prec*, arf_rnd_t *rnd*)

int **arf_mul_fmpz** (arf_t *z*, const arf_t *x*, const fmpz_t *y*, long *prec*, arf_rnd_t *rnd*)
Sets $z = x \times y$, rounded to *prec* bits in the direction specified by *rnd*, returning nonzero iff the operation is inexact.

int **arf_add** (arf_t *z*, const arf_t *x*, const arf_t *y*, long *prec*, arf_rnd_t *rnd*)

int **arf_add_si** (arf_t *z*, const arf_t *x*, long *y*, long *prec*, arf_rnd_t *rnd*) int **arf_add_ui** (arf_t *z*, const arf_t *x*, ulong *y*, long *prec*, arf_rnd_t *rnd*)

int **arf_add_fmpz** (arf_t *z*, const arf_t *x*, const fmpz_t *y*, long *prec*, arf_rnd_t *rnd*)
Sets $z = x + y$, rounded to *prec* bits in the direction specified by *rnd*, returning nonzero iff the operation is inexact.

int **arf_add_fmpz_2exp** (arf_t *z*, const arf_t *x*, const fmpz_t *y*, const fmpz_t *e*, long *prec*, arf_rnd_t *rnd*)
Sets $z = x + y2^e$, rounded to *prec* bits in the direction specified by *rnd*, returning nonzero iff the operation is inexact.

int **arf_sub** (arf_t *z*, const arf_t *x*, const arf_t *y*, long *prec*, arf_rnd_t *rnd*)

int **arf_sub_si** (arf_t *z*, const arf_t *x*, long *y*, long *prec*, arf_rnd_t *rnd*)

int **arf_sub_ui** (arf_t *z*, const arf_t *x*, ulong *y*, long *prec*, arf_rnd_t *rnd*)

int **arf_sub_fmpz** (arf_t *z*, const arf_t *x*, const fmpz_t *y*, long *prec*, arf_rnd_t *rnd*)
Sets $z = x - y$, rounded to *prec* bits in the direction specified by *rnd*, returning nonzero iff the operation is inexact.

int **arf_addmul** (arf_t *z*, const arf_t *x*, const arf_t *y*, long *prec*, arf_rnd_t *rnd*)

int **arf_addmul_ui** (arf_t *z*, const arf_t *x*, ulong *y*, long *prec*, arf_rnd_t *rnd*)

int **arf_addmul_si** (arf_t *z*, const arf_t *x*, long *y*, long *prec*, arf_rnd_t *rnd*)

int **arf_addmul_mpz** (arf_t *z*, const arf_t *x*, const mpz_t *y*, long *prec*, arf_rnd_t *rnd*)

int **arf_addmul_fmpz** (arf_t *z*, const arf_t *x*, const fmpz_t *y*, long *prec*, arf_rnd_t *rnd*)
Sets $z = z + x \times y$, rounded to *prec* bits in the direction specified by *rnd*, returning nonzero iff the operation is inexact.

int **arf_submul** (arf_t *z*, const arf_t *x*, const arf_t *y*, long *prec*, arf_rnd_t *rnd*)

int **arf_submul_ui** (arf_t *z*, const arf_t *x*, ulong *y*, long *prec*, arf_rnd_t *rnd*)

int **arf_submul_si** (arf_t *z*, const arf_t *x*, long *y*, long *prec*, arf_rnd_t *rnd*)

int **arf_submul_mpz** (arf_t *z*, const arf_t *x*, const mpz_t *y*, long *prec*, arf_rnd_t *rnd*)

int **arf_submul_fmpz** (arf_t *z*, const arf_t *x*, const fmpz_t *y*, long *prec*, arf_rnd_t *rnd*)
Sets $z = z - x \times y$, rounded to *prec* bits in the direction specified by *rnd*, returning nonzero iff the operation is inexact.

3.2.11 Summation

int **arf_sum** (arf_t *s*, arf_srcptr *terms*, long *len*, long *prec*, arf_rnd_t *rnd*)
Sets *s* to the sum of the array *terms* of length *len*, rounded to *prec* bits in the direction specified by *rnd*. The sum is computed as if done without any intermediate rounding error, with only a single rounding applied to the final result. Unlike repeated calls to `arf_add()` with infinite precision, this function does not overflow if the magnitudes of the terms are far apart. Warning: this function is implemented naively, and the running time is quadratic with respect to *len* in the worst case.

3.2.12 Division

int **arf_div** (arf_t *z*, const arf_t *x*, const arf_t *y*, long *prec*, arf_rnd_t *rnd*)

int **arf_div_ui** (arf_t *z*, const arf_t *x*, ulong *y*, long *prec*, arf_rnd_t *rnd*)

int **arf_ui_div** (arf_t *z*, ulong *x*, const arf_t *y*, long *prec*, arf_rnd_t *rnd*)

int **arf_div_si** (arf_t *z*, const arf_t *x*, long *y*, long *prec*, arf_rnd_t *rnd*) int **arf_si_div** (arf_t *z*, long *x*, const arf_t *y*, long *prec*, arf_rnd_t *rnd*)

int **arf_div_fmpz** (arf_t *z*, const arf_t *x*, const fmpz_t *y*, long *prec*, arf_rnd_t *rnd*)

int **arf_fmpz_div** (arf_t *z*, const fmpz_t *x*, const arf_t *y*, long *prec*, arf_rnd_t *rnd*)

int **arf_fmpz_div_fmpz** (arf_t *z*, const fmpz_t *x*, const fmpz_t *y*, long *prec*, arf_rnd_t *rnd*)
Sets $z = x/y$, rounded to *prec* bits in the direction specified by *rnd*, returning nonzero iff the operation is inexact. The result is NaN if *y* is zero.

3.2.13 Square roots

int **arf_sqrt** (arf_t *z*, const arf_t *x*, long *prec*, arf_rnd_t *rnd*)

int **arf_sqrt_ui** (arf_t *z*, ulong *x*, long *prec*, arf_rnd_t *rnd*)

int **arf_sqrt_fmpz** (arf_t *z*, const fmpz_t *x*, long *prec*, arf_rnd_t *rnd*)
Sets $z = \sqrt{x}$, rounded to *prec* bits in the direction specified by *rnd*, returning nonzero iff the operation is inexact. The result is NaN if *x* is negative.

int **arf_rsqrt** (arf_t *z*, const arf_t *x*, long *prec*, arf_rnd_t *rnd*)
Sets $z = 1/\sqrt{x}$, rounded to *prec* bits in the direction specified by *rnd*, returning nonzero iff the operation is inexact. The result is NaN if *x* is negative, and $+\infty$ if *x* is zero.

3.2.14 Complex arithmetic

int **arf_complex_mul** (arf_t *e*, arf_t *f*, const arf_t *a*, const arf_t *b*, const arf_t *c*, const arf_t *d*, long *prec*, arf_rnd_t *rnd*)

int **arf_complex_mul_fallback** (arf_t *e*, arf_t *f*, const arf_t *a*, const arf_t *b*, const arf_t *c*, const arf_t *d*, long *prec*, arf_rnd_t *rnd*)
Computes the complex product $e + fi = (a + bi)(c + di)$, rounding both e and f correctly to *prec* bits in the direction specified by *rnd*. The first bit in the return code indicates inexactness of e, and the second bit indicates inexactness of f.

If any of the components *a*, *b*, *c*, *d* is zero, two real multiplications and no additions are done. This convention is used even if any other part contains an infinity or NaN, and the behavior with infinite/NaN input is defined accordingly.

The *fallback* version is implemented naively, for testing purposes. No squaring optimization is implemented.

int **arf_complex_sqr** (arf_t *e*, arf_t *f*, const arf_t *a*, const arf_t *b*, long *prec*, arf_rnd_t *rnd*)
Computes the complex square $e + fi = (a + bi)^2$. This function has identical semantics to `arf_complex_mul()` (with $c = a, b = d$), but is faster.

3.3 arb.h – real numbers represented as floating-point balls

An `arb_t` represents a ball over the real numbers, that is, an interval $[m \pm r] \equiv [m - r, m + r]$ where the midpoint m and the radius r are (extended) real numbers and r is nonnegative (possibly infinite). The result of an (approximate) operation done on `arb_t` variables is a ball which contains the result of the (mathematically exact) operation applied to any choice of points in the input balls. In general, the output ball is not the smallest possible.

The precision parameter passed to each function roughly indicates the precision to which calculations on the midpoint are carried out (operations on the radius are always done using a fixed, small precision.)

For arithmetic operations, the precision parameter currently simply specifies the precision of the corresponding `arf_t` operation. In the future, the arithmetic might be made faster by incorporating sloppy rounding (typically equivalent to a loss of 1-2 bits of effective working precision) when the result is known to be inexact (while still propagating errors rigorously, of course). Arithmetic operations done on exact input with exactly representable output are always guaranteed to produce exact output.

For more complex operations, the precision parameter indicates a minimum working precision (algorithms might allocate extra internal precision to attempt to produce an output accurate to the requested number of bits, especially when the required precision can be estimated easily, but this is not generally required).

If the precision is increased and the inputs either are exact or are computed with increased accuracy as well, the output should converge proportionally, absent any bugs. The general intended strategy for using ball arithmetic is to add a few guard bits, and then repeat the calculation as necessary with an exponentially increasing number of guard bits (Ziv's strategy) until the result is exact enough for one's purposes (typically the first attempt will be successful).

The following balls with an infinite or NaN component are permitted, and may be returned as output from functions.

- The ball $[+\infty \pm c]$, where c is finite, represents the point at positive infinity. Such a ball can always be replaced by $[+\infty \pm 0]$ while preserving mathematical correctness (this is currently not done automatically by the library).
- The ball $[-\infty \pm c]$, where c is finite, represents the point at negative infinity. Such a ball can always be replaced by $[-\infty \pm 0]$ while preserving mathematical correctness (this is currently not done automatically by the library).
- The ball $[c \pm \infty]$, where c is finite or infinite, represents the whole extended real line $[-\infty, +\infty]$. Such a ball can always be replaced by $[0 \pm \infty]$ while preserving mathematical correctness (this is currently not done automatically by the library). Note that there is no way to represent a half-infinite interval such as $[0, \infty]$.
- The ball $[\mathrm{NaN} \pm c]$, where c is finite or infinite, represents an indeterminate value (the value could be any extended real number, or it could represent a function being evaluated outside its domain of definition, for example where the result would be complex). Such an indeterminate ball can always be replaced by $[\mathrm{NaN} \pm \infty]$ while preserving mathematical correctness (this is currently not done automatically by the library).

The `arb_t` type is almost identical semantically to the legacy `fmprb_t` type, but uses a more efficient internal representation. Whereas the midpoint and radius of an `fmprb_t` both have the same type, the `arb_t` type uses an `arf_t` for the midpoint and a `mag_t` for the radius. Code designed to manipulate the radius of an `fmprb_t` directly can be ported to the `arb_t` type by writing the radius to a temporary `arf_t` variable, manipulating that variable, and then converting back to the `mag_t` radius. Alternatively, `mag_t` methods can be used directly where available.

3.3.1 Types, macros and constants

arb_struct

arb_t

An `arb_struct` consists of an `arf_struct` (the midpoint) and a `mag_struct` (the radius). An `arb_t` is defined as an array of length one of type `arb_struct`, permitting an `arb_t` to be passed by reference.

arb_ptr

Alias for `arb_struct *`, used for vectors of numbers.

arb_srcptr

Alias for `const arb_struct *`, used for vectors of numbers when passed as constant input to functions.

arb_midref(x)

Macro returning a pointer to the midpoint of *x* as an `arf_t`.

arb_radref(x)

Macro returning a pointer to the radius of *x* as a `mag_t`.

3.3.2 Memory management

void **arb_init** (arb_t *x*)
Initializes the variable *x* for use. Its midpoint and radius are both set to zero.

void **arb_clear** (arb_t *x*)
Clears the variable *x*, freeing or recycling its allocated memory.

arb_ptr **_arb_vec_init** (long *n*)
Returns a pointer to an array of *n* initialized `arb_struct` entries.

void **_arb_vec_clear** (arb_ptr *v*, long *n*)
Clears an array of *n* initialized `arb_struct` entries.

void **arb_swap** (arb_t *x*, arb_t *y*)
Swaps *x* and *y* efficiently.

3.3.3 Assignment and rounding

void **arb_set_fmprb** (arb_t *y*, const fmprb_t *x*)

void **arb_get_fmprb** (fmprb_t *y*, const arb_t *x*)

void **arb_set** (arb_t *y*, const arb_t *x*)

void **arb_set_arf** (arb_t *y*, const arf_t *x*)

void **arb_set_si** (arb_t *y*, long *x*)

void **arb_set_ui** (arb_t *y*, ulong *x*)

void **arb_set_fmpz** (arb_t *y*, const fmpz_t *x*)
Sets *y* to the value of *x* without rounding.

void **arb_set_fmpz_2exp** (arb_t *y*, const fmpz_t *x*, const fmpz_t *e*)
Sets *y* to $x \cdot 2^e$.

void **arb_set_round** (arb_t *y*, const arb_t *x*, long *prec*)

void **arb_set_round_fmpz** (arb_t *y*, const fmpz_t *x*, long *prec*)
Sets *y* to the value of *x*, rounded to *prec* bits.

void **arb_set_round_fmpz_2exp** (arb_t *y*, const fmpz_t *x*, const fmpz_t *e*, long *prec*)
Sets *y* to $x \cdot 2^e$, rounded to *prec* bits.

void **arb_set_fmpq** (arb_t *y*, const fmpq_t *x*, long *prec*)
Sets *y* to the rational number *x*, rounded to *prec* bits.

int **arb_set_str** (arb_t *res*, const char * *inp*, long *prec*)
Sets *res* to the value specified by the human-readable string *inp*. The input may be a decimal floating-point literal, such as "25", "0.001", "7e+141" or "-31.4159e-1", and may also consist of two such literals separated by the symbol "+/-" and optionally enclosed in brackets, e.g. "[3.25 +/- 0.0001]", or simply "[+/- 10]" with an implicit zero midpoint. The output is rounded to *prec* bits, and if the binary-to-decimal conversion is inexact, the resulting error is added to the radius.

The symbols "inf" and "nan" are recognized (a nan midpoint results in an indeterminate interval, with infinite radius).

Returns 0 if successful and nonzero if unsuccessful. If unsuccessful, the result is set to an indeterminate interval.

char * **arb_get_str** (const arb_t *x*, long *n*, ulong *flags*)
Returns a nice human-readable representation of *x*, with at most *n* digits of the midpoint printed.

With default flags, the output can be parsed back with `arb_set_str()`, and this is guaranteed to produce an interval containing the original interval *x*.

By default, the output is rounded so that the value given for the midpoint is correct up to 1 ulp (unit in the last decimal place).

If *ARB_STR_MORE* is added to *flags*, more (possibly incorrect) digits may be printed.

If *ARB_STR_NO_RADIUS* is added to *flags*, the radius is not included in the output if at least 1 digit of the midpoint can be printed.

By adding a multiple *m* of *ARB_STR_CONDENSE* to *flags*, strings of more than three times *m* consecutive digits are condensed, only printing the leading and trailing *m* digits along with brackets indicating the number of digits omitted (useful when computing values to extremely high precision).

3.3.4 Assignment of special values

void **arb_zero** (arb_t *x*)
Sets *x* to zero.

void **arb_one** (arb_t *f*)
Sets *x* to the exact integer 1.

void **arb_pos_inf** (arb_t *x*)
Sets *x* to positive infinity, with a zero radius.

void **arb_neg_inf** (arb_t *x*)
Sets *x* to negative infinity, with a zero radius.

void **arb_zero_pm_inf** (arb_t *x*)
Sets *x* to $[0 \pm \infty]$, representing the whole extended real line.

void **arb_indeterminate** (arb_t *x*)
Sets *x* to $[\mathrm{NaN} \pm \infty]$, representing an indeterminate result.

3.3.5 Input and output

void **arb_print** (const arb_t *x*)
Prints the internal representation of *x*.

void **arb_printd** (const arb_t *x*, long *digits*)
Prints *x* in decimal. The printed value of the radius is not adjusted to compensate for the fact that the binary-to-decimal conversion of both the midpoint and the radius introduces additional error.

void **arb_printn** (const arb_t *x*, long *digits*, ulong *flags*)
Prints a nice decimal representation of *x*. By default, the output is guaranteed to be correct to within one unit in the last digit. An error bound is also printed explicitly. See `arb_get_str()` for details.

3.3.6 Random number generation

void **arb_randtest** (arb_t *x*, flint_rand_t *state*, long *prec*, long *mag_bits*)
Generates a random ball. The midpoint and radius will both be finite.

void **arb_randtest_exact** (arb_t *x*, flint_rand_t *state*, long *prec*, long *mag_bits*)
Generates a random number with zero radius.

void **arb_randtest_precise** (arb_t *x*, flint_rand_t *state*, long *prec*, long *mag_bits*)
Generates a random number with radius around $2^{-\mathrm{prec}}$ the magnitude of the midpoint.

void **arb_randtest_wide** (arb_t *x*, flint_rand_t *state*, long *prec*, long *mag_bits*)
Generates a random number with midpoint and radius chosen independently, possibly giving a very large interval.

void **arb_randtest_special** (arb_t *x*, flint_rand_t *state*, long *prec*, long *mag_bits*)
Generates a random interval, possibly having NaN or an infinity as the midpoint and possibly having an infinite radius.

void **arb_get_rand_fmpq** (fmpq_t *q*, flint_rand_t *state*, const arb_t *x*, long *bits*)
Sets *q* to a random rational number from the interval represented by *x*. A denominator is chosen by multiplying the binary denominator of *x* by a random integer up to *bits* bits.

The outcome is undefined if the midpoint or radius of *x* is non-finite, or if the exponent of the midpoint or radius is so large or small that representing the endpoints as exact rational numbers would cause overflows.

3.3.7 Radius and interval operations

void **arb_add_error_arf** (arb_t *x*, const arf_t *err*)
Adds *err*, which is assumed to be nonnegative, to the radius of *x*.

void **arb_add_error_2exp_si** (arb_t *x*, long *e*)

void **arb_add_error_2exp_fmpz** (arb_t *x*, const fmpz_t *e*)
Adds 2^e to the radius of *x*.

void **arb_add_error** (arb_t *x*, const arb_t *error*)
Adds the supremum of *err*, which is assumed to be nonnegative, to the radius of *x*.

void **arb_union** (arb_t *z*, const arb_t *x*, const arb_t *y*, long *prec*)
Sets *z* to a ball containing both *x* and *y*.

void **arb_get_abs_ubound_arf** (arf_t *u*, const arb_t *x*, long *prec*)
Sets *u* to the upper bound for the absolute value of *x*, rounded up to *prec* bits. If *x* contains NaN, the result is NaN.

void **arb_get_abs_lbound_arf** (arf_t *u*, const arb_t *x*, long *prec*)
Sets *u* to the lower bound for the absolute value of *x*, rounded down to *prec* bits. If *x* contains NaN, the result is NaN.

void **arb_get_mag** (mag_t *z*, const arb_t *x*)
Sets *z* to an upper bound for the absolute value of *x*. If *x* contains NaN, the result is positive infinity.

void **arb_get_mag_lower** (mag_t *z*, const arb_t *x*)
Sets *z* to a lower bound for the absolute value of *x*. If *x* contains NaN, the result is zero.

arb_get_mag_lower_nonnegative (mag_t *z*, const arb_t *x*)
Sets *z* to a lower bound for the signed value of *x*, or zero if *x* overlaps with the negative half-axis. If *x* contains NaN, the result is zero.

void **arb_get_interval_fmpz_2exp** (fmpz_t *a*, fmpz_t *b*, fmpz_t *exp*, const arb_t *x*)
Computes the exact interval represented by *x*, in the form of an integer interval multiplied by a power of two, i.e. $x = [a, b] \times 2^{\mathrm{exp}}$.

The outcome is undefined if the midpoint or radius of *x* is non-finite, or if the difference in magnitude between the midpoint and radius is so large that representing the endpoints exactly would cause overflows.

void **arb_set_interval_arf** (arb_t *x*, const arf_t *a*, const arf_t *b*, long *prec*)

void **arb_set_interval_mpfr** (arb_t *x*, const mpfr_t *a*, const mpfr_t *b*, long *prec*)
Sets *x* to a ball containing the interval $[a, b]$. We require that $a \leq b$.

void **arb_get_interval_arf** (arf_t *a*, arf_t *b*, const arb_t *x*, long *prec*)

void **arb_get_interval_mpfr** (mpfr_t *a*, mpfr_t *b*, const arb_t *x*)
Constructs an interval $[a, b]$ containing the ball *x*. The MPFR version uses the precision of the output variables.

long **arb_rel_error_bits** (const arb_t *x*)
Returns the effective relative error of *x* measured in bits, defined as the difference between the position of the top bit in the radius and the top bit in the midpoint, plus one. The result is clamped between plus/minus *ARF_PREC_EXACT*.

long **arb_rel_accuracy_bits** (const arb_t *x*)
Returns the effective relative accuracy of *x* measured in bits, equal to the negative of the return value from `arb_rel_error_bits()`.

long **arb_bits** (const arb_t *x*)
Returns the number of bits needed to represent the absolute value of the mantissa of the midpoint of *x*, i.e. the minimum precision sufficient to represent *x* exactly. Returns 0 if the midpoint of *x* is a special value.

void **arb_trim** (arb_t *y*, const arb_t *x*)
Sets *y* to a trimmed copy of *x*: rounds *x* to a number of bits equal to the accuracy of *x* (as indicated by its radius), plus a few guard bits. The resulting ball is guaranteed to contain *x*, but is more economical if *x* has less than full accuracy.

int **arb_get_unique_fmpz** (fmpz_t *z*, const arb_t *x*)
If *x* contains a unique integer, sets *z* to that value and returns nonzero. Otherwise (if *x* represents no integers or more than one integer), returns zero.

void **arb_floor** (arb_t *y*, const arb_t *x*, long *prec*)

void **arb_ceil** (arb_t *y*, const arb_t *x*, long *prec*)
Sets *y* to a ball containing $\lfloor x \rfloor$ and $\lceil x \rceil$ respectively, with the midpoint of *y* rounded to at most *prec* bits.

void **arb_get_fmpz_mid_rad_10exp** (fmpz_t *mid*, fmpz_t *rad*, fmpz_t *exp*, const arb_t *x*, long *n*)
Assuming that *x* is finite and not exactly zero, computes integers *mid*, *rad*, *exp* such that $x \in [m-r, m+r] \times 10^e$ and such that the larger out of *mid* and *rad* has at least *n* digits plus a few guard digits. If *x* is infinite or exactly zero, the outputs are all set to zero.

3.3.8 Comparisons

int **arb_is_zero** (const arb_t *x*)
Returns nonzero iff the midpoint and radius of *x* are both zero.

int **arb_is_nonzero** (const arb_t *x*)
Returns nonzero iff zero is not contained in the interval represented by *x*.

int **arb_is_one** (const arb_t *f*)
Returns nonzero iff *x* is exactly 1.

int **arb_is_finite** (const arb_t *x*)
Returns nonzero iff the midpoint and radius of *x* are both finite floating-point numbers, i.e. not infinities or NaN.

int **arb_is_exact** (const arb_t *x*)
Returns nonzero iff the radius of *x* is zero.

int **arb_is_int** (const arb_t *x*)
Returns nonzero iff *x* is an exact integer.

int **arb_equal** (const arb_t *x*, const arb_t *y*)
Returns nonzero iff *x* and *y* are equal as balls, i.e. have both the same midpoint and radius.

Note that this is not the same thing as testing whether both *x* and *y* certainly represent the same real number, unless either *x* or *y* is exact (and neither contains NaN). To test whether both operands *might* represent the same mathematical quantity, use `arb_overlaps()` or `arb_contains()`, depending on the circumstance.

int **arb_is_positive** (const arb_t *x*)

int **arb_is_nonnegative** (const arb_t *x*)

int **arb_is_negative** (const arb_t *x*)

int **arb_is_nonpositive** (const arb_t *x*)
Returns nonzero iff all points *p* in the interval represented by *x* satisfy, respectively, $p > 0$, $p \geq 0$, $p < 0$, $p \leq 0$. If *x* contains NaN, returns zero.

int **arb_overlaps** (const arb_t *x*, const arb_t *y*)
Returns nonzero iff *x* and *y* have some point in common. If either *x* or *y* contains NaN, this function always returns nonzero (as a NaN could be anything, it could in particular contain any number that is included in the other operand).

int **arb_contains_arf** (const arb_t *x*, const arf_t *y*)

int **arb_contains_fmpq** (const arb_t *x*, const fmpq_t *y*)

int **arb_contains_fmpz** (const arb_t *x*, const fmpz_t *y*)

int **arb_contains_si** (const arb_t *x*, long *y*)

int **arb_contains_mpfr** (const arb_t *x*, const mpfr_t *y*)

int **arb_contains** (const arb_t *x*, const arb_t *y*)
Returns nonzero iff the given number (or ball) *y* is contained in the interval represented by *x*.

If *x* is contains NaN, this function always returns nonzero (as it could represent anything, and in particular could represent all the points included in *y*). If *y* contains NaN and *x* does not, it always returns zero.

int **arb_contains_zero** (const arb_t *x*)

int **arb_contains_negative** (const arb_t *x*)

int **arb_contains_nonpositive** (const arb_t *x*)

int **arb_contains_positive** (const arb_t *x*)

int **arb_contains_nonnegative** (const arb_t *x*)
Returns nonzero iff there is any point *p* in the interval represented by *x* satisfying, respectively, $p = 0$, $p < 0$, $p \leq 0$, $p > 0$, $p \geq 0$. If *x* contains NaN, returns nonzero.

3.3.9 Arithmetic

void **arb_neg** (arb_t *y*, const arb_t *x*)

void **arb_neg_round** (arb_t *y*, const arb_t *x*, long *prec*)
Sets *y* to the negation of *x*.

void **arb_abs** (arb_t *x*, const arb_t *y*)
Sets *y* to the absolute value of *x*. No attempt is made to improve the interval represented by *x* if it contains zero.

void **arb_add** (arb_t *z*, const arb_t *x*, const arb_t *y*, long *prec*)

void **arb_add_arf** (arb_t *z*, const arb_t *x*, const arf_t *y*, long *prec*)

void **arb_add_ui** (arb_t *z*, const arb_t *x*, ulong *y*, long *prec*)

void **arb_add_si** (arb_t *z*, const arb_t *x*, long *y*, long *prec*)

void **arb_add_fmpz** (arb_t *z*, const arb_t *x*, const fmpz_t *y*, long *prec*)
Sets $z = x + y$, rounded to *prec* bits. The precision can be *ARF_PREC_EXACT* provided that the result fits in memory.

void **arb_add_fmpz_2exp** (arb_t *z*, const arb_t *x*, const fmpz_t *m*, const fmpz_t *e*, long *prec*)
Sets $z = x + m \cdot 2^e$, rounded to *prec* bits. The precision can be *ARF_PREC_EXACT* provided that the result fits in memory.

void **arb_sub** (arb_t *z*, const arb_t *x*, const arb_t *y*, long *prec*)

void **arb_sub_arf** (arb_t *z*, const arb_t *x*, const arf_t *y*, long *prec*)

void **arb_sub_ui** (arb_t *z*, const arb_t *x*, ulong *y*, long *prec*)

void **arb_sub_si** (arb_t *z*, const arb_t *x*, long *y*, long *prec*)

void **arb_sub_fmpz** (arb_t *z*, const arb_t *x*, const fmpz_t *y*, long *prec*)
Sets $z = x - y$, rounded to *prec* bits. The precision can be *ARF_PREC_EXACT* provided that the result fits in memory.

void **arb_mul** (arb_t *z*, const arb_t *x*, const arb_t *y*, long *prec*)

void **arb_mul_arf** (arb_t *z*, const arb_t *x*, const arf_t *y*, long *prec*)

void **arb_mul_si** (arb_t *z*, const arb_t *x*, long *y*, long *prec*)

void **arb_mul_ui** (arb_t *z*, const arb_t *x*, ulong *y*, long *prec*)

void **arb_mul_fmpz** (arb_t *z*, const arb_t *x*, const fmpz_t *y*, long *prec*)
Sets $z = x \cdot y$, rounded to *prec* bits. The precision can be *ARF_PREC_EXACT* provided that the result fits in memory.

void **arb_mul_2exp_si** (arb_t *y*, const arb_t *x*, long *e*)

void **arb_mul_2exp_fmpz** (arb_t *y*, const arb_t *x*, const fmpz_t *e*)
Sets *y* to *x* multiplied by 2^e.

void **arb_addmul** (arb_t *z*, const arb_t *x*, const arb_t *y*, long *prec*)

void **arb_addmul_arf** (arb_t *z*, const arb_t *x*, const arf_t *y*, long *prec*)

void **arb_addmul_si** (arb_t *z*, const arb_t *x*, long *y*, long *prec*)

void **arb_addmul_ui** (arb_t *z*, const arb_t *x*, ulong *y*, long *prec*)

void **arb_addmul_fmpz** (arb_t *z*, const arb_t *x*, const fmpz_t *y*, long *prec*)
Sets $z = z + x \cdot y$, rounded to prec bits. The precision can be *ARF_PREC_EXACT* provided that the result fits in memory.

void **arb_submul** (arb_t *z*, const arb_t *x*, const arb_t *y*, long *prec*)

void **arb_submul_arf** (arb_t *z*, const arb_t *x*, const arf_t *y*, long *prec*)

void **arb_submul_si** (arb_t *z*, const arb_t *x*, long *y*, long *prec*)

void **arb_submul_ui** (arb_t *z*, const arb_t *x*, ulong *y*, long *prec*)

void **arb_submul_fmpz** (arb_t *z*, const arb_t *x*, const fmpz_t *y*, long *prec*)
Sets $z = z - x \cdot y$, rounded to prec bits. The precision can be *ARF_PREC_EXACT* provided that the result fits in memory.

void **arb_inv** (arb_t *y*, const arb_t *x*, long *prec*)
Sets *z* to $1/x$.

void **arb_div** (arb_t *z*, const arb_t *x*, const arb_t *y*, long *prec*)

void **arb_div_arf** (arb_t *z*, const arb_t *x*, const arf_t *y*, long *prec*)

void **arb_div_si** (arb_t *z*, const arb_t *x*, long *y*, long *prec*)

void **arb_div_ui** (arb_t *z*, const arb_t *x*, ulong *y*, long *prec*) void **arb_div_fmpz** (arb_t *z*, const arb_t *x*, const fmpz_t *y*, long *prec*)

void **arb_fmpz_div_fmpz** (arb_t *z*, const fmpz_t *x*, const fmpz_t *y*, long *prec*)

void **arb_ui_div** (arb_t *z*, ulong *x*, const arb_t *y*, long *prec*)

Sets $z = x/y$, rounded to *prec* bits. If *y* contains zero, *z* is set to $0 \pm \infty$. Otherwise, error propagation uses the rule

$$\left| \frac{x}{y} - \frac{x+\xi_1 a}{y+\xi_2 b} \right| = \left| \frac{x\xi_2 b - y\xi_1 a}{y(y+\xi_2 b)} \right| \le \frac{|xb|+|ya|}{|y|(|y|-b)}$$

where $-1 \le \xi_1, \xi_2 \le 1$, and where the triangle inequality has been applied to the numerator and the reverse triangle inequality has been applied to the denominator.

void **arb_div_2expm1_ui** (arb_t *z*, const arb_t *x*, ulong *n*, long *prec*)

Sets $z = x/(2^n - 1)$, rounded to *prec* bits.

3.3.10 Powers and roots

void **arb_sqrt** (arb_t *z*, const arb_t *x*, long *prec*)

void **arb_sqrt_arf** (arb_t *z*, const arf_t *x*, long *prec*)

void **arb_sqrt_fmpz** (arb_t *z*, const fmpz_t *x*, long *prec*)

void **arb_sqrt_ui** (arb_t *z*, ulong *x*, long *prec*)

Sets *z* to the square root of *x*, rounded to *prec* bits.

If $x = m \pm x$ where $m \ge r \ge 0$, the propagated error is bounded by $\sqrt{m} - \sqrt{m-r} = \sqrt{m}(1 - \sqrt{1-r/m}) \le \sqrt{m}(r/m + (r/m)^2)/2$.

void **arb_sqrtpos** (arb_t *z*, const arb_t *x*, long *prec*)

Sets *z* to the square root of *x*, assuming that *x* represents a nonnegative number (i.e. discarding any negative numbers in the input interval), and producing an output interval not containing any negative numbers (unless the radius is infinite).

void **arb_hypot** (arb_t *z*, const arb_t *x*, const arb_t *y*, long *prec*)

Sets *z* to $\sqrt{x^2 + y^2}$.

void **arb_rsqrt** (arb_t *z*, const arb_t *x*, long *prec*)

void **arb_rsqrt_ui** (arb_t *z*, ulong *x*, long *prec*)

Sets *z* to the reciprocal square root of *x*, rounded to *prec* bits. At high precision, this is faster than computing a square root.

void **arb_sqrt1pm1** (arb_t *z*, const arb_t *x*, long *prec*)

Sets $z = \sqrt{1+x} - 1$, computed accurately when $x \approx 0$.

void **arb_root** (arb_t *z*, const arb_t *x*, ulong *k*, long *prec*)

Sets *z* to the *k*-th root of *x*, rounded to *prec* bits. As currently implemented, this function is only fast for small *k*. For large *k* it is better to use `arb_pow_fmpq()` or `arb_pow()`.

void **arb_pow_fmpz_binexp** (arb_t *y*, const arb_t *b*, const fmpz_t *e*, long *prec*)

void **arb_pow_fmpz** (arb_t *y*, const arb_t *b*, const fmpz_t *e*, long *prec*)

void **arb_pow_ui** (arb_t *y*, const arb_t *b*, ulong *e*, long *prec*)

void **arb_ui_pow_ui** (arb_t *y*, ulong *b*, ulong *e*, long *prec*)

void **arb_si_pow_ui** (arb_t *y*, long *b*, ulong *e*, long *prec*)

Sets $y = b^e$ using binary exponentiation (with an initial division if $e < 0$). Provided that *b* and *e* are small enough and the exponent is positive, the exact power can be computed by setting the precision to *ARF_PREC_EXACT*.

Note that these functions can get slow if the exponent is extremely large (in such cases `arb_pow()` may be superior).

void **arb_pow_fmpq**(arb_t *y*, const arb_t *x*, const fmpq_t *a*, long *prec*)
Sets $y = b^e$, computed as $y = (b^{1/q})^p$ if the denominator of $e = p/q$ is small, and generally as $y = \exp(e \log b)$.

Note that this function can get slow if the exponent is extremely large (in such cases `arb_pow()` may be superior).

void **arb_pow**(arb_t *z*, const arb_t *x*, const arb_t *y*, long *prec*)
Sets $z = x^y$, computed using binary exponentiation if y if a small exact integer, as $z = (x^{1/2})^{2y}$ if y is a small exact half-integer, and generally as $z = \exp(y \log x)$.

3.3.11 Exponentials and logarithms

void **arb_log_ui**(arb_t *z*, ulong *x*, long *prec*)

void **arb_log_fmpz**(arb_t *z*, const fmpz_t *x*, long *prec*)

void **arb_log_arf**(arb_t *z*, const arf_t *x*, long *prec*)

void **arb_log**(arb_t *z*, const arb_t *x*, long *prec*)
Sets $z = \log(x)$.

At low to medium precision (up to about 4096 bits), `arb_log_arf()` uses table-based argument reduction and fast Taylor series evaluation via `_arb_atan_taylor_rs()`. At high precision, it falls back to MPFR. The function `arb_log()` simply calls `arb_log_arf()` with the midpoint as input, and separately adds the propagated error.

void **arb_log_ui_from_prev**(arb_t *log_k1*, ulong *k1*, arb_t *log_k0*, ulong *k0*, long *prec*)
Computes $\log(k_1)$, given $\log(k_0)$ where $k_0 < k_1$. At high precision, this function uses the formula $\log(k_1) = \log(k_0) + 2 \operatorname{atanh}((k_1 - k_0)/(k_1 + k_0))$, evaluating the inverse hyperbolic tangent using binary splitting (for best efficiency, k_0 should be large and $k_1 - k_0$ should be small). Otherwise, it ignores $\log(k_0)$ and evaluates the logarithm the usual way.

void **arb_log1p**(arb_t *z*, const arb_t *x*, long *prec*)
Sets $z = \log(1 + x)$, computed accurately when $x \approx 0$.

void **arb_exp**(arb_t *z*, const arb_t *x*, long *prec*)
Sets $z = \exp(x)$. Error propagation is done using the following rule: assuming $x = m \pm r$, the error is largest at $m + r$, and we have $\exp(m + r) - \exp(m) = \exp(m)(\exp(r) - 1) \le r \exp(m + r)$.

void **arb_expm1**(arb_t *z*, const arb_t *x*, long *prec*)
Sets $z = \exp(x) - 1$, computed accurately when $x \approx 0$.

3.3.12 Trigonometric functions

void **arb_sin**(arb_t *s*, const arb_t *x*, long *prec*)

void **arb_cos**(arb_t *c*, const arb_t *x*, long *prec*)

void **arb_sin_cos**(arb_t *s*, arb_t *c*, const arb_t *x*, long *prec*)
Sets $s = \sin(x)$, $c = \cos(x)$. Error propagation uses the rule $|\sin(m \pm r) - \sin(m)| \le \min(r, 2)$.

void **arb_sin_pi**(arb_t *s*, const arb_t *x*, long *prec*)

void **arb_cos_pi**(arb_t *c*, const arb_t *x*, long *prec*)

void **arb_sin_cos_pi**(arb_t *s*, arb_t *c*, const arb_t *x*, long *prec*)
Sets $s = \sin(\pi x)$, $c = \cos(\pi x)$.

void **arb_tan** (arb_t *y*, const arb_t *x*, long *prec*)
Sets $y = \tan(x) = \sin(x)/\cos(y)$.

void **arb_cot** (arb_t *y*, const arb_t *x*, long *prec*)
Sets $y = \cot(x) = \cos(x)/\sin(y)$.

void **arb_sin_cos_pi_fmpq** (arb_t *s*, arb_t *c*, const fmpq_t *x*, long *prec*)

void **arb_sin_pi_fmpq** (arb_t *s*, const fmpq_t *x*, long *prec*)

void **arb_cos_pi_fmpq** (arb_t *c*, const fmpq_t *x*, long *prec*)
Sets $s = \sin(\pi x)$, $c = \cos(\pi x)$ where x is a rational number (whose numerator and denominator are assumed to be reduced). We first use trigonometric symmetries to reduce the argument to the octant $[0, 1/4]$. Then we either multiply by a numerical approximation of π and evaluate the trigonometric function the usual way, or we use algebraic methods, depending on which is estimated to be faster. Since the argument has been reduced to the first octant, the first of these two methods gives full accuracy even if the original argument is close to some root other the origin.

void **arb_tan_pi** (arb_t *y*, const arb_t *x*, long *prec*)
Sets $y = \tan(\pi x)$.

void **arb_cot_pi** (arb_t *y*, const arb_t *x*, long *prec*)
Sets $y = \cot(\pi x)$.

3.3.13 Inverse trigonometric functions

void **arb_atan_arf** (arb_t *z*, const arf_t *x*, long *prec*)

void **arb_atan** (arb_t *z*, const arb_t *x*, long *prec*)
Sets $z = \operatorname{atan}(x)$.

At low to medium precision (up to about 4096 bits), `arb_atan_arf()` uses table-based argument reduction and fast Taylor series evaluation via `_arb_atan_taylor_rs()`. At high precision, it falls back to MPFR. The function `arb_atan()` simply calls `arb_atan_arf()` with the midpoint as input, and separately adds the propagated error.

The function `arb_atan_arf()` uses lookup tables if possible, and otherwise falls back to `arb_atan_arf_bb()`.

void **arb_atan2** (arb_t *z*, const arb_t *b*, const arb_t *a*, long *prec*)
Sets *r* to an the argument (phase) of the complex number $a + bi$, with the branch cut discontinuity on $(-\infty, 0]$. We define $\operatorname{atan2}(0, 0) = 0$, and for $a < 0$, $\operatorname{atan2}(0, a) = \pi$.

void **arb_asin** (arb_t *z*, const arb_t *x*, long *prec*)
Sets $z = \operatorname{asin}(x) = \operatorname{atan}(x/\sqrt{1 - x^2})$. If x is not contained in the domain $[-1, 1]$, the result is an indeterminate interval.

void **arb_acos** (arb_t *z*, const arb_t *x*, long *prec*)
Sets $z = \operatorname{acos}(x) = \pi/2 - \operatorname{asin}(x)$. If x is not contained in the domain $[-1, 1]$, the result is an indeterminate interval.

3.3.14 Hyperbolic functions

void **arb_sinh** (arb_t *s*, const arb_t *x*, long *prec*)

void **arb_cosh** (arb_t *c*, const arb_t *x*, long *prec*)

void **arb_sinh_cosh** (arb_t *s*, arb_t *c*, const arb_t *x*, long *prec*)
Sets $s = \sinh(x)$, $c = \cosh(x)$. If the midpoint of x is close to zero and the hyperbolic sine is to be computed, evaluates $(e^{2x} \pm 1)/(2e^x)$ via `arb_expm1()` to avoid loss of accuracy. Otherwise evaluates $(e^x \pm e^{-x})/2$.

void **arb_tanh** (arb_t *y*, const arb_t *x*, long *prec*)
Sets $y = \tanh(x) = \sinh(x)/\cosh(x)$, evaluated via `arb_expm1()` as $\tanh(x) = (e^{2x} - 1)/(e^{2x} + 1)$ if the midpoint of x is negative and as $\tanh(x) = (1 - e^{-2x})/(1 + e^{-2x})$ otherwise.

void **arb_coth** (arb_t *y*, const arb_t *x*, long *prec*)
Sets $y = \coth(x) = \cosh(x)/\sinh(x)$, evaluated using the same strategy as `arb_tanh()`.

3.3.15 Inverse hyperbolic functions

void **arb_atanh** (arb_t *z*, const arb_t *x*, long *prec*)
Sets $z = \operatorname{atanh}(x)$.

void **arb_asinh** (arb_t *z*, const arb_t *x*, long *prec*)
Sets $z = \operatorname{asinh}(x)$.

void **arb_acosh** (arb_t *z*, const arb_t *x*, long *prec*)
Sets $z = \operatorname{acosh}(x)$. If $x < 1$, the result is an indeterminate interval.

3.3.16 Constants

The following functions cache the computed values to speed up repeated calls at the same or lower precision. For further implementation details, see *Algorithms for mathematical constants*.

void **arb_const_pi** (arb_t *z*, long *prec*)
Computes π.

void **arb_const_sqrt_pi** (arb_t *z*, long *prec*)
Computes $\sqrt{\pi}$.

void **arb_const_log_sqrt2pi** (arb_t *z*, long *prec*)
Computes $\log \sqrt{2\pi}$.

void **arb_const_log2** (arb_t *z*, long *prec*)
Computes $\log(2)$.

void **arb_const_log10** (arb_t *z*, long *prec*)
Computes $\log(10)$.

void **arb_const_euler** (arb_t *z*, long *prec*)
Computes Euler's constant $\gamma = \lim_{k\to\infty}(H_k - \log k)$ where $H_k = 1 + 1/2 + \ldots + 1/k$.

void **arb_const_catalan** (arb_t *z*, long *prec*)
Computes Catalan's constant $C = \sum_{n=0}^{\infty}(-1)^n/(2n+1)^2$.

void **arb_const_e** (arb_t *z*, long *prec*)
Computes $e = \exp(1)$.

void **arb_const_khinchin** (arb_t *z*, long *prec*)
Computes Khinchin's constant K_0.

void **arb_const_glaisher** (arb_t *z*, long *prec*)
Computes the Glaisher-Kinkelin constant $A = \exp(1/12 - \zeta'(-1))$.

void **arb_const_apery** (arb_t *z*, long *prec*)
Computes Apery's constant $\zeta(3)$.

3.3.17 Gamma function and factorials

void **arb_rising_ui_bs** (arb_t *z*, const arb_t *x*, ulong *n*, long *prec*)

void **arb_rising_ui_rs** (arb_t *z*, const arb_t *x*, ulong *n*, ulong *step*, long *prec*)

void **arb_rising_ui_rec** (arb_t *z*, const arb_t *x*, ulong *n*, long *prec*)

void **arb_rising_ui** (arb_t *z*, const arb_t *x*, ulong *n*, long *prec*)

Computes the rising factorial $z = x(x+1)(x+2)\cdots(x+n-1)$.

The *bs* version uses binary splitting. The *rs* version uses rectangular splitting. The *rec* version uses either *bs* or *rs* depending on the input. The default version is currently identical to the *rec* version. In a future version, it will use the gamma function or asymptotic series when this is more efficient.

The *rs* version takes an optional *step* parameter for tuning purposes (to use the default step length, pass zero).

void **arb_rising_fmpq_ui** (arb_t *z*, const fmpq_t *x*, ulong *n*, long *prec*)

Computes the rising factorial $z = x(x+1)(x+2)\cdots(x+n-1)$ using binary splitting. If the denominator or numerator of *x* is large compared to *prec*, it is more efficient to convert *x* to an approximation and use `arb_rising_ui()`.

void **arb_rising2_ui_bs** (arb_t *u*, arb_t *v*, const arb_t *x*, ulong *n*, long *prec*)

void **arb_rising2_ui_rs** (arb_t *u*, arb_t *v*, const arb_t *x*, ulong *n*, ulong *step*, long *prec*)

void **arb_rising2_ui** (arb_t *u*, arb_t *v*, const arb_t *x*, ulong *n*, long *prec*)

Letting $u(x) = x(x+1)(x+2)\cdots(x+n-1)$, simultaneously compute $u(x)$ and $v(x) = u'(x)$, respectively using binary splitting, rectangular splitting (with optional nonzero step length *step* to override the default choice), and an automatic algorithm choice.

void **arb_fac_ui** (arb_t *z*, ulong *n*, long *prec*)

Computes the factorial $z = n!$ via the gamma function.

void **arb_bin_ui** (arb_t *z*, const arb_t *n*, ulong *k*, long *prec*)

void **arb_bin_uiui** (arb_t *z*, ulong *n*, ulong *k*, long *prec*)

Computes the binomial coefficient $z = \binom{n}{k}$, via the rising factorial as $\binom{n}{k} = (n-k+1)_k/k!$.

void **arb_gamma** (arb_t *z*, const arb_t *x*, long *prec*)

void **arb_gamma_fmpq** (arb_t *z*, const fmpq_t *x*, long *prec*)

void **arb_gamma_fmpz** (arb_t *z*, const fmpz_t *x*, long *prec*)

Computes the gamma function $z = \Gamma(x)$.

void **arb_lgamma** (arb_t *z*, const arb_t *x*, long *prec*)

Computes the logarithmic gamma function $z = \log \Gamma(x)$. The complex branch structure is assumed, so if $x \le 0$, the result is an indeterminate interval.

void **arb_rgamma** (arb_t *z*, const arb_t *x*, long *prec*)

Computes the reciprocal gamma function $z = 1/\Gamma(x)$, avoiding division by zero at the poles of the gamma function.

void **arb_digamma** (arb_t *y*, const arb_t *x*, long *prec*)

Computes the digamma function $z = \psi(x) = (\log \Gamma(x))' = \Gamma'(x)/\Gamma(x)$.

3.3.18 Zeta function

void **arb_zeta_ui_vec_borwein** (arb_ptr *z*, ulong *start*, long *num*, ulong *step*, long *prec*)

Evaluates $\zeta(s)$ at num consecutive integers *s* beginning with *start* and proceeding in increments of *step*. Uses

Borwein's formula ([Bor2000], [GS2003]), implemented to support fast multi-evaluation (but also works well for a single s).

Requires $\mathrm{start} \geq 2$. For efficiency, the largest s should be at most about as large as *prec*. Arguments approaching *LONG_MAX* will cause overflows. One should therefore only use this function for s up to about *prec*, and then switch to the Euler product.

The algorithm for single s is basically identical to the one used in MPFR (see [MPFR2012] for a detailed description). In particular, we evaluate the sum backwards to avoid storing more than one d_k coefficient, and use integer arithmetic throughout since it is convenient and the terms turn out to be slightly larger than 2^{prec}. The only numerical error in the main loop comes from the division by k^s, which adds less than 1 unit of error per term. For fast multi-evaluation, we repeatedly divide by k^{step}. Each division reduces the input error and adds at most 1 unit of additional rounding error, so by induction, the error per term is always smaller than 2 units.

void **arb_zeta_ui_asymp** (arb_t *x*, ulong *s*, long *prec*)
Assuming $s \geq 2$, approximates $\zeta(s)$ by $1 + 2^{-s}$ along with a correct error bound. We use the following bounds: for $s > b$, $\zeta(s) - 1 < 2^{-b}$, and generally, $\zeta(s) - (1 + 2^{-s}) < 2^{2 - \lfloor 3s/2 \rfloor}$.

void **arb_zeta_ui_euler_product** (arb_t *z*, ulong *s*, long *prec*)
Computes $\zeta(s)$ using the Euler product. This is fast only if s is large compared to the precision.

Writing $P(a, b) = \prod_{a \leq p \leq b} (1 - p^{-s})$, we have $1/\zeta(s) = P(a, M) P(M + 1, \infty)$.

To bound the error caused by truncating the product at M, we write $P(M + 1, \infty) = 1 - \epsilon(s, M)$. Since $0 < P(a, M) \leq 1$, the absolute error for $\zeta(s)$ is bounded by $\epsilon(s, M)$.

According to the analysis in [Fil1992], it holds for all $s \geq 6$ and $M \geq 1$ that $1/P(M + 1, \infty) - 1 \leq f(s, M) \equiv 2M^{1-s}/(s/2 - 1)$. Thus, we have $1/(1 - \epsilon(s, M)) - 1 \leq f(s, M)$, and expanding the geometric series allows us to conclude that $\epsilon(M) \leq f(s, M)$.

void **arb_zeta_ui_bernoulli** (arb_t *x*, ulong *s*, long *prec*)
Computes $\zeta(s)$ for even s via the corresponding Bernoulli number.

void **arb_zeta_ui_borwein_bsplit** (arb_t *x*, ulong *s*, long *prec*)
Computes $\zeta(s)$ for arbitrary $s \geq 2$ using a binary splitting implementation of Borwein's algorithm. This has quasilinear complexity with respect to the precision (assuming that s is fixed).

void **arb_zeta_ui_vec** (arb_ptr *x*, ulong *start*, long *num*, long *prec*)

void **arb_zeta_ui_vec_even** (arb_ptr *x*, ulong *start*, long *num*, long *prec*)

void **arb_zeta_ui_vec_odd** (arb_ptr *x*, ulong *start*, long *num*, long *prec*)
Computes $\zeta(s)$ at *num* consecutive integers (respectively *num* even or *num* odd integers) beginning with $s = \mathrm{start} \geq 2$, automatically choosing an appropriate algorithm.

void **arb_zeta_ui** (arb_t *x*, ulong *s*, long *prec*)
Computes $\zeta(s)$ for nonnegative integer $s \neq 1$, automatically choosing an appropriate algorithm. This function is intended for numerical evaluation of isolated zeta values; for multi-evaluation, the vector versions are more efficient.

void **arb_zeta** (arb_t *z*, const arb_t *s*, long *prec*)
Sets *z* to the value of the Riemann zeta function $\zeta(s)$.

For computing derivatives with respect to s, use `arb_poly_zeta_series()`.

void **arb_hurwitz_zeta** (arb_t *z*, const arb_t *s*, const arb_t *a*, long *prec*)
Sets *z* to the value of the Hurwitz zeta function $\zeta(s, a)$.

For computing derivatives with respect to s, use `arb_poly_zeta_series()`.

3.3.19 Bernoulli numbers

void **arb_bernoulli_ui** (arb_t *b*, ulong *n*, long *prec*)

Sets b to the numerical value of the Bernoulli number B_n accurate to *prec* bits, computed by a division of the exact fraction if B_n is in the global cache or the exact numerator roughly is larger than *prec* bits, and using `arb_bernoulli_ui_zeta()` otherwise. This function reads B_n from the global cache if the number is already cached, but does not automatically extend the cache by itself.

void **arb_bernoulli_ui_zeta** (arb_t *b*, ulong *n*, long *prec*)

Sets b to the numerical value of B_n accurate to *prec* bits, computed using the formula $B_{2n} = (-1)^{n+1} 2 (2n)! \zeta(2n) / (2\pi)^n$.

To avoid potential infinite recursion, we explicitly call the Euler product implementation of the zeta function. We therefore assume that the precision is small enough and n large enough for the Euler product to converge rapidly (otherwise this function will effectively hang).

3.3.20 Polylogarithms

void **arb_polylog** (arb_t *w*, const arb_t *s*, const arb_t *z*, long *prec*)

void **arb_polylog_si** (arb_t *w*, long *s*, const arb_t *z*, long *prec*)

Sets *w* to the polylogarithm $\mathrm{Li}_s(z)$.

3.3.21 Other special functions

void **arb_fib_fmpz** (arb_t *z*, const fmpz_t *n*, long *prec*)

void **arb_fib_ui** (arb_t *z*, ulong *n*, long *prec*)

Computes the Fibonacci number F_n. Uses the binary squaring algorithm described in [Tak2000]. Provided that *n* is small enough, an exact Fibonacci number can be computed by setting the precision to *ARF_PREC_EXACT*.

void **arb_agm** (arb_t *z*, const arb_t *x*, const arb_t *y*, long *prec*)

Sets *z* to the arithmetic-geometric mean of *x* and *y*.

void **arb_chebyshev_t_ui** (arb_t *a*, ulong *n*, const arb_t *x*, long *prec*)

void **arb_chebyshev_u_ui** (arb_t *a*, ulong *n*, const arb_t *x*, long *prec*)

Evaluates the Chebyshev polynomial of the first kind $a = T_n(x)$ or the Chebyshev polynomial of the second kind $a = U_n(x)$.

void **arb_chebyshev_t2_ui** (arb_t *a*, arb_t *b*, ulong *n*, const arb_t *x*, long *prec*)

void **arb_chebyshev_u2_ui** (arb_t *a*, arb_t *b*, ulong *n*, const arb_t *x*, long *prec*)

Simultaneously evaluates $a = T_n(x), b = T_{n-1}(x)$ or $a = U_n(x), b = U_{n-1}(x)$. Aliasing between *a*, *b* and *x* is not permitted.

3.3.22 Internals for computing elementary functions

void **_arb_atan_taylor_naive** (mp_ptr *y*, mp_limb_t * *error*, mp_srcptr *x*, mp_size_t *xn*, ulong *N*, int *alternating*)

void **_arb_atan_taylor_rs** (mp_ptr *y*, mp_limb_t * *error*, mp_srcptr *x*, mp_size_t *xn*, ulong *N*, int *alternating*)

Computes an approximation of $y = \sum_{k=0}^{N-1} x^{2k+1}/(2k+1)$ (if *alternating* is 0) or $y = \sum_{k=0}^{N-1} (-1)^k x^{2k+1}/(2k+1)$ (if *alternating* is 1). Used internally for computing arctangents and logarithms.

The *naive* version uses the forward recurrence, and the *rs* version uses a division-avoiding rectangular splitting scheme.

Requires $N \le 255$, $0 \le x \le 1/16$, and *xn* positive. The input *x* and output *y* are fixed-point numbers with *xn* fractional limbs. A bound for the ulp error is written to *error*.

void **`_arb_exp_taylor_naive`**(mp_ptr *y*, mp_limb_t * *error*, mp_srcptr *x*, mp_size_t *xn*, ulong *N*)

void **`_arb_exp_taylor_rs`**(mp_ptr *y*, mp_limb_t * *error*, mp_srcptr *x*, mp_size_t *xn*, ulong *N*)

Computes an approximation of $y = \sum_{k=0}^{N-1} x^k/k!$. Used internally for computing exponentials. The *naive* version uses the forward recurrence, and the *rs* version uses a division-avoiding rectangular splitting scheme.

Requires $N \le 287$, $0 \le x \le 1/16$, and *xn* positive. The input *x* is a fixed-point number with *xn* fractional limbs, and the output *y* is a fixed-point number with *xn* fractional limbs plus one extra limb for the integer part of the result.

A bound for the ulp error is written to *error*.

void **`_arb_sin_cos_taylor_naive`**(mp_ptr *ysin*, mp_ptr *ycos*, mp_limb_t * *error*, mp_srcptr *x*, mp_size_t *xn*, ulong *N*)

void **`_arb_sin_cos_taylor_rs`**(mp_ptr *ysin*, mp_ptr *ycos*, mp_limb_t * *error*, mp_srcptr *x*, mp_size_t *xn*, ulong *N*, int *sinonly*, int *alternating*)

Computes approximations of $y_s = \sum_{k=0}^{N-1}(-1)^k x^{2k+1}/(2k+1)!$ and $y_c = \sum_{k=0}^{N-1}(-1)^k x^{2k}/(2k)!$. Used internally for computing sines and cosines. The *naive* version uses the forward recurrence, and the *rs* version uses a division-avoiding rectangular splitting scheme.

Requires $N \le 143$, $0 \le x \le 1/16$, and *xn* positive. The input *x* and outputs *ysin*, *ycos* are fixed-point numbers with *xn* fractional limbs. A bound for the ulp error is written to *error*.

If *sinonly* is 1, only the sine is computed; if *sinonly* is 0 both the sine and cosine are computed. To compute sin and cos, *alternating* should be 1. If *alternating* is 0, the hyperbolic sine is computed (this is currently only intended to be used together with *sinonly*).

int **`_arb_get_mpn_fixed_mod_log2`**(mp_ptr *w*, fmpz_t *q*, mp_limb_t * *error*, const arf_t *x*, mp_size_t *wn*)

Attempts to write $w = x - q\log(2)$ with $0 \le w < \log(2)$, where *w* is a fixed-point number with *wn* limbs and ulp error *error*. Returns success.

int **`_arb_get_mpn_fixed_mod_pi4`**(mp_ptr *w*, fmpz_t *q*, int * *octant*, mp_limb_t * *error*, const arf_t *x*, mp_size_t *wn*)

Attempts to write $w = |x| - q\pi/4$ with $0 \le w < \pi/4$, where *w* is a fixed-point number with *wn* limbs and ulp error *error*. Returns success.

The value of *q* mod 8 is written to *octant*. The output variable *q* can be NULL, in which case the full value of *q* is not stored.

long **`_arb_exp_taylor_bound`**(long *mag*, long *prec*)

Returns *n* such that $\left|\sum_{k=n}^{\infty} x^k/k!\right| \le 2^{-\mathrm{prec}}$, assuming $|x| \le 2^{\mathrm{mag}} \le 1/4$.

void **`arb_exp_arf_bb`**(arb_t *z*, const arf_t *x*, long *prec*, int *m1*)

Computes the exponential function using the bit-burst algorithm. If *m1* is nonzero, the exponential function minus one is computed accurately.

Aborts if *x* is extremely small or large (where another algorithm should be used).

For large *x*, repeated halving is used. In fact, we always do argument reduction until $|x|$ is smaller than about 2^{-d} where $d \approx 16$ to speed up convergence. If $|x| \approx 2^m$, we thus need about $m + d$ squarings.

Computing $\log(2)$ costs roughly 100-200 multiplications, so is not usually worth the effort at very high precision. However, this function could be improved by using $\log(2)$ based reduction at precision low enough that the value can be assumed to be cached.

void **_arb_exp_sum_bs_simple** (fmpz_t *T*, fmpz_t *Q*, mp_bitcnt_t * *Qexp*, const fmpz_t *x*, mp_bitcnt_t *r*, long *N*)

void **_arb_exp_sum_bs_powtab** (fmpz_t *T*, fmpz_t *Q*, mp_bitcnt_t * *Qexp*, const fmpz_t *x*, mp_bitcnt_t *r*, long *N*)

Computes *T*, *Q* and *Qexp* such that $T/(Q2^{\text{Qexp}}) = \sum_{k=1}^{N}(x/2^r)^k/k!$ using binary splitting. Note that the sum is taken to *N* inclusive and omits the constant term.

The *powtab* version precomputes a table of powers of *x*, resulting in slightly higher memory usage but better speed. For best efficiency, *N* should have many trailing zero bits.

void **_arb_atan_sum_bs_simple** (fmpz_t *T*, fmpz_t *Q*, mp_bitcnt_t * *Qexp*, const fmpz_t *x*, mp_bitcnt_t *r*, long *N*)

void **_arb_atan_sum_bs_powtab** (fmpz_t *T*, fmpz_t *Q*, mp_bitcnt_t * *Qexp*, const fmpz_t *x*, mp_bitcnt_t *r*, long *N*)

Computes *T*, *Q* and *Qexp* such that $T/(Q2^{\text{Qexp}}) = \sum_{k=1}^{N}(-1)^k(x/2^r)^{2k}/(2k+1)$ using binary splitting. Note that the sum is taken to *N* inclusive, omits the linear term, and requires a final multiplication by $(x/2^r)$ to give the true series for atan.

The *powtab* version precomputes a table of powers of *x*, resulting in slightly higher memory usage but better speed. For best efficiency, *N* should have many trailing zero bits.

void **arb_atan_arf_bb** (arb_t *z*, const arf_t *x*, long *prec*)

Computes the arctangent of *x*. Initially, the argument-halving formula

$$\operatorname{atan}(x) = 2\operatorname{atan}\left(\frac{x}{1+\sqrt{1+x^2}}\right)$$

is applied up to 8 times to get a small argument. Then a version of the bit-burst algorithm is used. The functional equation

$$\operatorname{atan}(x) = \operatorname{atan}(p/q) + \operatorname{atan}(w), \quad w = \frac{qx-p}{px+q}, \quad p = \lfloor qx \rfloor$$

is applied repeatedly instead of integrating a differential equation for the arctangent, as this appears to be more efficient.

3.4 arb_poly.h – polynomials over the real numbers

An `arb_poly_t` represents a polynomial over the real numbers, implemented as an array of coefficients of type `arb_struct`.

Most functions are provided in two versions: an underscore method which operates directly on pre-allocated arrays of coefficients and generally has some restrictions (such as requiring the lengths to be nonzero and not supporting aliasing of the input and output arrays), and a non-underscore method which performs automatic memory management and handles degenerate cases.

3.4.1 Types, macros and constants

arb_poly_struct

arb_poly_t

Contains a pointer to an array of coefficients (coeffs), the used length (length), and the allocated size of the array (alloc).

An *arb_poly_t* is defined as an array of length one of type *arb_poly_struct*, permitting an *arb_poly_t* to be passed by reference.

3.4.2 Memory management

void **arb_poly_init** (arb_poly_t *poly*)
Initializes the polynomial for use, setting it to the zero polynomial.

void **arb_poly_clear** (arb_poly_t *poly*)
Clears the polynomial, deallocating all coefficients and the coefficient array.

void **arb_poly_fit_length** (arb_poly_t *poly*, long *len*)
Makes sures that the coefficient array of the polynomial contains at least *len* initialized coefficients.

void **_arb_poly_set_length** (arb_poly_t *poly*, long *len*)
Directly changes the length of the polynomial, without allocating or deallocating coefficients. The value shold not exceed the allocation length.

void **_arb_poly_normalise** (arb_poly_t *poly*)
Strips any trailing coefficients which are identical to zero.

3.4.3 Basic manipulation

void **arb_poly_zero** (arb_poly_t *poly*)

void **arb_poly_one** (arb_poly_t *poly*)
Sets *poly* to the constant 0 respectively 1.

void **arb_poly_set** (arb_poly_t *dest*, const arb_poly_t *src*)
Sets *dest* to a copy of *src*.

void **arb_poly_set_round** (arb_poly_t *dest*, const arb_poly_t *src*, long *prec*)
Sets *dest* to a copy of *src*, rounded to *prec* bits.

void **arb_poly_set_coeff_si** (arb_poly_t *poly*, long *n*, long *c*)

void **arb_poly_set_coeff_arb** (arb_poly_t *poly*, long *n*, const arb_t *c*)
Sets the coefficient with index *n* in *poly* to the value *c*. We require that *n* is nonnegative.

void **arb_poly_get_coeff_arb** (arb_t *v*, const arb_poly_t *poly*, long *n*)
Sets *v* to the value of the coefficient with index *n* in *poly*. We require that *n* is nonnegative.

arb_poly_get_coeff_ptr (poly, n)
Given $n \geq 0$, returns a pointer to coefficient *n* of *poly*, or *NULL* if *n* exceeds the length of *poly*.

void **_arb_poly_shift_right** (arb_ptr *res*, arb_srcptr *poly*, long *len*, long *n*)

void **arb_poly_shift_right** (arb_poly_t *res*, const arb_poly_t *poly*, long *n*)
Sets *res* to *poly* divided by x^n, throwing away the lower coefficients. We require that *n* is nonnegative.

void **_arb_poly_shift_left** (arb_ptr *res*, arb_srcptr *poly*, long *len*, long *n*)

void **arb_poly_shift_left** (arb_poly_t *res*, const arb_poly_t *poly*, long *n*)
Sets *res* to *poly* multiplied by x^n. We require that *n* is nonnegative.

void **arb_poly_truncate** (arb_poly_t *poly*, long *n*)
Truncates *poly* to have length at most *n*, i.e. degree strictly smaller than *n*.

long **arb_poly_length** (const arb_poly_t *poly*)
Returns the length of *poly*, i.e. zero if *poly* is identically zero, and otherwise one more than the index of the highest term that is not identically zero.

long **arb_poly_degree** (const arb_poly_t *poly*)
Returns the degree of *poly*, defined as one less than its length. Note that if one or several leading coefficients are

balls containing zero, this value can be larger than the true degree of the exact polynomial represented by *poly*, so the return value of this function is effectively an upper bound.

3.4.4 Conversions

void **arb_poly_set_fmpz_poly** (arb_poly_t *poly*, const fmpz_poly_t *src*, long *prec*)

void **arb_poly_set_fmpq_poly** (arb_poly_t *poly*, const fmpq_poly_t *src*, long *prec*)

void **arb_poly_set_si** (arb_poly_t *poly*, long *src*)
Sets *poly* to *src*, rounding the coefficients to *prec* bits.

3.4.5 Input and output

void **arb_poly_printd** (const arb_poly_t *poly*, long *digits*)
Prints the polynomial as an array of coefficients, printing each coefficient using *arb_printd*.

3.4.6 Random generation

void **arb_poly_randtest** (arb_poly_t *poly*, flint_rand_t *state*, long *len*, long *prec*, long *mag_bits*)
Creates a random polynomial with length at most *len*.

3.4.7 Comparisons

int **arb_poly_contains** (const arb_poly_t *poly1*, const arb_poly_t *poly2*)

int **arb_poly_contains_fmpz_poly** (const arb_poly_t *poly1*, const fmpz_poly_t *poly2*)

int **arb_poly_contains_fmpq_poly** (const arb_poly_t *poly1*, const fmpq_poly_t *poly2*)
Returns nonzero iff *poly1* contains *poly2*.

int **arb_poly_equal** (const arb_poly_t *A*, const arb_poly_t *B*)
Returns nonzero iff *A* and *B* are equal as polynomial balls, i.e. all coefficients have equal midpoint and radius.

int **_arb_poly_overlaps** (arb_srcptr *poly1*, long *len1*, arb_srcptr *poly2*, long *len2*)

int **arb_poly_overlaps** (const arb_poly_t *poly1*, const arb_poly_t *poly2*)
Returns nonzero iff *poly1* overlaps with *poly2*. The underscore function requires that *len1* ist at least as large as *len2*.

int **arb_poly_get_unique_fmpz_poly** (fmpz_poly_t *z*, const arb_poly_t *x*)
If *x* contains a unique integer polynomial, sets *z* to that value and returns nonzero. Otherwise (if *x* represents no integers or more than one integer), returns zero, possibly partially modifying *z*.

3.4.8 Arithmetic

void **_arb_poly_add** (arb_ptr *C*, arb_srcptr *A*, long *lenA*, arb_srcptr *B*, long *lenB*, long *prec*)
Sets *{C, max(lenA, lenB)}* to the sum of *{A, lenA}* and *{B, lenB}*. Allows aliasing of the input and output operands.

void **arb_poly_add** (arb_poly_t *C*, const arb_poly_t *A*, const arb_poly_t *B*, long *prec*)
Sets *C* to the sum of *A* and *B*.

void **_arb_poly_sub** (arb_ptr *C*, arb_srcptr *A*, long *lenA*, arb_srcptr *B*, long *lenB*, long *prec*)
Sets *{C, max(lenA, lenB)}* to the difference of *{A, lenA}* and *{B, lenB}*. Allows aliasing of the input and output operands.

void **arb_poly_sub** (arb_poly_t *C*, const arb_poly_t *A*, const arb_poly_t *B*, long *prec*)
Sets *C* to the difference of *A* and *B*.

void **arb_poly_neg** (arb_poly_t *C*, const arb_poly_t *A*)
Sets *C* to the negation of *A*.

void **arb_poly_scalar_mul_2exp_si** (arb_poly_t *C*, const arb_poly_t *A*, long *c*)
Sets *C* to *A* multiplied by 2^c.

void **_arb_poly_mullow_classical** (arb_ptr *C*, arb_srcptr *A*, long *lenA*, arb_srcptr *B*, long *lenB*, long *n*, long *prec*)

void **_arb_poly_mullow_block** (arb_ptr *C*, arb_srcptr *A*, long *lenA*, arb_srcptr *B*, long *lenB*, long *n*, long *prec*)

void **_arb_poly_mullow** (arb_ptr *C*, arb_srcptr *A*, long *lenA*, arb_srcptr *B*, long *lenB*, long *n*, long *prec*)
Sets *{C, n}* to the product of *{A, lenA}* and *{B, lenB}*, truncated to length *n*. The output is not allowed to be aliased with either of the inputs. We require $\mathrm{lenA} \ge \mathrm{lenB} > 0$, $n > 0$, $\mathrm{lenA} + \mathrm{lenB} - 1 \ge n$.

The *classical* version uses a plain loop. This has good numerical stability but gets slow for large *n*.

The *block* version decomposes the product into several subproducts which are computed exactly over the integers.

It first attempts to find an integer c such that $A(2^c x)$ and $B(2^c x)$ have slowly varying coefficients, to reduce the number of blocks.

The scaling factor c is chosen in a quick, heuristic way by picking the first and last nonzero terms in each polynomial. If the indices in A are a_2, a_1 and the log-2 magnitudes are e_2, e_1, and the indices in B are b_2, b_1 with corresponding magnitudes f_2, f_1, then we compute c as the weighted arithmetic mean of the slopes, rounded to the nearest integer:

$$c = \left\lfloor \frac{(e_2 - e_1) + (f_2 + f_1)}{(a_2 - a_1) + (b_2 - b_1)} + \frac{1}{2} \right\rfloor .$$

This strategy is used because it is simple. It is not optimal in all cases, but will typically give good performance when multiplying two power series with a similar decay rate.

The default algorithm chooses the *classical* algorithm for short polynomials and the *block* algorithm for long polynomials.

If the input pointers are identical (and the lengths are the same), they are assumed to represent the same polynomial, and its square is computed.

void **arb_poly_mullow_classical** (arb_poly_t *C*, const arb_poly_t *A*, const arb_poly_t *B*, long *n*, long *prec*)

void **arb_poly_mullow_ztrunc** (arb_poly_t *C*, const arb_poly_t *A*, const arb_poly_t *B*, long *n*, long *prec*)

void **arb_poly_mullow_block** (arb_poly_t *C*, const arb_poly_t *A*, const arb_poly_t *B*, long *n*, long *prec*)

void **arb_poly_mullow** (arb_poly_t *C*, const arb_poly_t *A*, const arb_poly_t *B*, long *n*, long *prec*)
Sets *C* to the product of *A* and *B*, truncated to length *n*. If the same variable is passed for *A* and *B*, sets *C* to the square of *A* truncated to length *n*.

void **_arb_poly_mul** (arb_ptr *C*, arb_srcptr *A*, long *lenA*, arb_srcptr *B*, long *lenB*, long *prec*)
Sets *{C, lenA + lenB - 1}* to the product of *{A, lenA}* and *{B, lenB}*. The output is not allowed to be aliased with either of the inputs. We require $\mathrm{lenA} \ge \mathrm{lenB} > 0$. This function is implemented as a simple wrapper for `_arb_poly_mullow()`.

If the input pointers are identical (and the lengths are the same), they are assumed to represent the same polynomial, and its square is computed.

void **arb_poly_mul** (arb_poly_t *C*, const arb_poly_t *A*, const arb_poly_t *B*, long *prec*)
Sets *C* to the product of *A* and *B*. If the same variable is passed for *A* and *B*, sets *C* to the square of *A*.

void **_arb_poly_inv_series** (arb_ptr *Q*, arb_srcptr *A*, long *Alen*, long *len*, long *prec*)
Sets *{Q, len}* to the power series inverse of *{A, Alen}*. Uses Newton iteration.

void **arb_poly_inv_series** (arb_poly_t *Q*, const arb_poly_t *A*, long *n*, long *prec*)
Sets *Q* to the power series inverse of *A*, truncated to length *n*.

void **_arb_poly_div_series** (arb_ptr *Q*, arb_srcptr *A*, long *Alen*, arb_srcptr *B*, long *Blen*, long *n*, long *prec*)
Sets *{Q, n}* to the power series quotient of *{A, Alen}* by *{B, Blen}*. Uses Newton iteration followed by multiplication.

void **arb_poly_div_series** (arb_poly_t *Q*, const arb_poly_t *A*, const arb_poly_t *B*, long *n*, long *prec*)
Sets *Q* to the power series quotient *A* divided by *B*, truncated to length *n*.

void **_arb_poly_div** (arb_ptr *Q*, arb_srcptr *A*, long *lenA*, arb_srcptr *B*, long *lenB*, long *prec*)

void **_arb_poly_rem** (arb_ptr *R*, arb_srcptr *A*, long *lenA*, arb_srcptr *B*, long *lenB*, long *prec*)

void **_arb_poly_divrem** (arb_ptr *Q*, arb_ptr *R*, arb_srcptr *A*, long *lenA*, arb_srcptr *B*, long *lenB*, long *prec*)

int **arb_poly_divrem** (arb_poly_t *Q*, arb_poly_t *R*, const arb_poly_t *A*, const arb_poly_t *B*, long *prec*)
Performs polynomial division with remainder, computing a quotient Q and a remainder R such that $A = BQ + R$. The implementation reverses the inputs and performs power series division.

If the leading coefficient of B contains zero (or if B is identically zero), returns 0 indicating failure without modifying the outputs. Otherwise returns nonzero.

void **_arb_poly_div_root** (arb_ptr *Q*, arb_t *R*, arb_srcptr *A*, long *len*, const arb_t *c*, long *prec*)
Divides A by the polynomial $x - c$, computing the quotient Q as well as the remainder $R = f(c)$.

3.4.9 Composition

void **_arb_poly_compose_horner** (arb_ptr *res*, arb_srcptr *poly1*, long *len1*, arb_srcptr *poly2*, long *len2*, long *prec*)

void **arb_poly_compose_horner** (arb_poly_t *res*, const arb_poly_t *poly1*, const arb_poly_t *poly2*, long *prec*)

void **_arb_poly_compose_divconquer** (arb_ptr *res*, arb_srcptr *poly1*, long *len1*, arb_srcptr *poly2*, long *len2*, long *prec*)

void **arb_poly_compose_divconquer** (arb_poly_t *res*, const arb_poly_t *poly1*, const arb_poly_t *poly2*, long *prec*)

void **_arb_poly_compose** (arb_ptr *res*, arb_srcptr *poly1*, long *len1*, arb_srcptr *poly2*, long *len2*, long *prec*)

void **arb_poly_compose** (arb_poly_t *res*, const arb_poly_t *poly1*, const arb_poly_t *poly2*, long *prec*)
Sets *res* to the composition $h(x) = f(g(x))$ where f is given by *poly1* and g is given by *poly2*, respectively using Horner's rule, divide-and-conquer, and an automatic choice between the two algorithms. The underscore methods do not support aliasing of the output with either input polynomial.

void **_arb_poly_compose_series_horner** (arb_ptr *res*, arb_srcptr *poly1*, long *len1*, arb_srcptr *poly2*, long *len2*, long *n*, long *prec*)

void **arb_poly_compose_series_horner** (arb_poly_t *res*, const arb_poly_t *poly1*, const arb_poly_t *poly2*, long *n*, long *prec*) void **_arb_poly_compose_series_brent_kung** (arb_ptr *res*, arb_srcptr *poly1*, long *len1*, arb_srcptr *poly2*, long *len2*, long *n*, long *prec*)

void **arb_poly_compose_series_brent_kung** (arb_poly_t *res*, const arb_poly_t *poly1*, const arb_poly_t *poly2*, long *n*, long *prec*)

void **_arb_poly_compose_series** (arb_ptr *res*, arb_srcptr *poly1*, long *len1*, arb_srcptr *poly2*, long *len2*, long *n*, long *prec*)

void **arb_poly_compose_series** (arb_poly_t *res*, const arb_poly_t *poly1*, const arb_poly_t *poly2*, long *n*, long *prec*)

Sets *res* to the power series composition $h(x) = f(g(x))$ truncated to order $O(x^n)$ where f is given by *poly1* and g is given by *poly2*, respectively using Horner's rule, the Brent-Kung baby step-giant step algorithm, and an automatic choice between the two algorithms. We require that the constant term in $g(x)$ is exactly zero. The underscore methods do not support aliasing of the output with either input polynomial.

void **_arb_poly_revert_series_lagrange** (arb_ptr *h*, arb_srcptr *f*, long *flen*, long *n*, long *prec*)

void **arb_poly_revert_series_lagrange** (arb_poly_t *h*, const arb_poly_t *f*, long *n*, long *prec*)

void **_arb_poly_revert_series_newton** (arb_ptr *h*, arb_srcptr *f*, long *flen*, long *n*, long *prec*)

void **arb_poly_revert_series_newton** (arb_poly_t *h*, const arb_poly_t *f*, long *n*, long *prec*)

void **_arb_poly_revert_series_lagrange_fast** (arb_ptr *h*, arb_srcptr *f*, long *flen*, long *n*, long *prec*)

void **arb_poly_revert_series_lagrange_fast** (arb_poly_t *h*, const arb_poly_t *f*, long *n*, long *prec*)

void **_arb_poly_revert_series** (arb_ptr *h*, arb_srcptr *f*, long *flen*, long *n*, long *prec*)

void **arb_poly_revert_series** (arb_poly_t *h*, const arb_poly_t *f*, long *n*, long *prec*)

Sets h to the power series reversion of f, i.e. the expansion of the compositional inverse function $f^{-1}(x)$, truncated to order $O(x^n)$, using respectively Lagrange inversion, Newton iteration, fast Lagrange inversion, and a default algorithm choice.

We require that the constant term in f is exactly zero and that the linear term is nonzero. The underscore methods assume that *flen* is at least 2, and do not support aliasing.

3.4.10 Evaluation

void **_arb_poly_evaluate_horner** (arb_t *y*, arb_srcptr *f*, long *len*, const arb_t *x*, long *prec*)

void **arb_poly_evaluate_horner** (arb_t *y*, const arb_poly_t *f*, const arb_t *x*, long *prec*)

void **_arb_poly_evaluate_rectangular** (arb_t *y*, arb_srcptr *f*, long *len*, const arb_t *x*, long *prec*)

void **arb_poly_evaluate_rectangular** (arb_t *y*, const arb_poly_t *f*, const arb_t *x*, long *prec*)

void **_arb_poly_evaluate** (arb_t *y*, arb_srcptr *f*, long *len*, const arb_t *x*, long *prec*)

void **arb_poly_evaluate** (arb_t *y*, const arb_poly_t *f*, const arb_t *x*, long *prec*)

Sets $y = f(x)$, evaluated respectively using Horner's rule, rectangular splitting, and an automatic algorithm choice.

void **_arb_poly_evaluate_acb_horner** (acb_t *y*, arb_srcptr *f*, long *len*, const acb_t *x*, long *prec*)

void **arb_poly_evaluate_acb_horner** (acb_t *y*, const arb_poly_t *f*, const acb_t *x*, long *prec*)

void **_arb_poly_evaluate_acb_rectangular** (acb_t *y*, arb_srcptr *f*, long *len*, const acb_t *x*, long *prec*)

void **arb_poly_evaluate_acb_rectangular** (acb_t *y*, const arb_poly_t *f*, const acb_t *x*, long *prec*)

void **_arb_poly_evaluate_acb** (acb_t *y*, arb_srcptr *f*, long *len*, const acb_t *x*, long *prec*)

void **arb_poly_evaluate_acb** (acb_t *y*, const arb_poly_t *f*, const acb_t *x*, long *prec*)
Sets $y = f(x)$ where x is a complex number, evaluating the polynomial respectively using Horner's rule, rectangular splitting, and an automatic algorithm choice.

void **_arb_poly_evaluate2_horner** (arb_t *y*, arb_t *z*, arb_srcptr *f*, long *len*, const arb_t *x*, long *prec*)

void **arb_poly_evaluate2_horner** (arb_t *y*, arb_t *z*, const arb_poly_t *f*, const arb_t *x*, long *prec*)

void **_arb_poly_evaluate2_rectangular** (arb_t *y*, arb_t *z*, arb_srcptr *f*, long *len*, const arb_t *x*, long *prec*)

void **arb_poly_evaluate2_rectangular** (arb_t *y*, arb_t *z*, const arb_poly_t *f*, const arb_t *x*, long *prec*)

void **_arb_poly_evaluate2** (arb_t *y*, arb_t *z*, arb_srcptr *f*, long *len*, const arb_t *x*, long *prec*)

void **arb_poly_evaluate2** (arb_t *y*, arb_t *z*, const arb_poly_t *f*, const arb_t *x*, long *prec*)
Sets $y = f(x), z = f'(x)$, evaluated respectively using Horner's rule, rectangular splitting, and an automatic algorithm choice.

When Horner's rule is used, the only advantage of evaluating the function and its derivative simultaneously is that one does not have to generate the derivative polynomial explicitly. With the rectangular splitting algorithm, the powers can be reused, making simultaneous evaluation slightly faster.

void **_arb_poly_evaluate2_acb_horner** (acb_t *y*, acb_t *z*, arb_srcptr *f*, long *len*, const acb_t *x*, long *prec*)

void **arb_poly_evaluate2_acb_horner** (acb_t *y*, acb_t *z*, const arb_poly_t *f*, const acb_t *x*, long *prec*)

void **_arb_poly_evaluate2_acb_rectangular** (acb_t *y*, acb_t *z*, arb_srcptr *f*, long *len*, const acb_t *x*, long *prec*)

void **arb_poly_evaluate2_acb_rectangular** (acb_t *y*, acb_t *z*, const arb_poly_t *f*, const acb_t *x*, long *prec*)

void **_arb_poly_evaluate2_acb** (acb_t *y*, acb_t *z*, arb_srcptr *f*, long *len*, const acb_t *x*, long *prec*)

void **arb_poly_evaluate2_acb** (acb_t *y*, acb_t *z*, const arb_poly_t *f*, const acb_t *x*, long *prec*)
Sets $y = f(x), z = f'(x)$, evaluated respectively using Horner's rule, rectangular splitting, and an automatic algorithm choice.

3.4.11 Product trees

void **_arb_poly_product_roots** (arb_ptr *poly*, arb_srcptr *xs*, long *n*, long *prec*)

void **arb_poly_product_roots** (arb_poly_t *poly*, arb_srcptr *xs*, long *n*, long *prec*)
Generates the polynomial $(x - x_0)(x - x_1) \cdots (x - x_{n-1})$.

arb_ptr * **_arb_poly_tree_alloc** (long *len*)
Returns an initialized data structured capable of representing a remainder tree (product tree) of *len* roots.

void **_arb_poly_tree_free** (arb_ptr * *tree*, long *len*)
Deallocates a tree structure as allocated using *_arb_poly_tree_alloc*.

void **_arb_poly_tree_build** (arb_ptr * *tree*, arb_srcptr *roots*, long *len*, long *prec*)
Constructs a product tree from a given array of *len* roots. The tree structure must be pre-allocated to the specified length using `_arb_poly_tree_alloc()`.

3.4.12 Multipoint evaluation

void **_arb_poly_evaluate_vec_iter** (arb_ptr *ys*, arb_srcptr *poly*, long *plen*, arb_srcptr *xs*, long *n*, long *prec*)

void **arb_poly_evaluate_vec_iter** (arb_ptr *ys*, const arb_poly_t *poly*, arb_srcptr *xs*, long *n*, long *prec*)
Evaluates the polynomial simultaneously at *n* given points, calling `_arb_poly_evaluate()` repeatedly.

void **_arb_poly_evaluate_vec_fast_precomp** (arb_ptr *vs*, arb_srcptr *poly*, long *plen*, arb_ptr * *tree*, long *len*, long *prec*)

void **_arb_poly_evaluate_vec_fast** (arb_ptr *ys*, arb_srcptr *poly*, long *plen*, arb_srcptr *xs*, long *n*, long *prec*)

void **arb_poly_evaluate_vec_fast** (arb_ptr *ys*, const arb_poly_t *poly*, arb_srcptr *xs*, long *n*, long *prec*)
Evaluates the polynomial simultaneously at *n* given points, using fast multipoint evaluation.

3.4.13 Interpolation

void **_arb_poly_interpolate_newton** (arb_ptr *poly*, arb_srcptr *xs*, arb_srcptr *ys*, long *n*, long *prec*)

void **arb_poly_interpolate_newton** (arb_poly_t *poly*, arb_srcptr *xs*, arb_srcptr *ys*, long *n*, long *prec*)
Recovers the unique polynomial of length at most *n* that interpolates the given *x* and *y* values. This implementation first interpolates in the Newton basis and then converts back to the monomial basis.

void **_arb_poly_interpolate_barycentric** (arb_ptr *poly*, arb_srcptr *xs*, arb_srcptr *ys*, long *n*, long *prec*)

void **arb_poly_interpolate_barycentric** (arb_poly_t *poly*, arb_srcptr *xs*, arb_srcptr *ys*, long *n*, long *prec*)
Recovers the unique polynomial of length at most *n* that interpolates the given *x* and *y* values. This implementation uses the barycentric form of Lagrange interpolation.

void **_arb_poly_interpolation_weights** (arb_ptr *w*, arb_ptr * *tree*, long *len*, long *prec*)

void **_arb_poly_interpolate_fast_precomp** (arb_ptr *poly*, arb_srcptr *ys*, arb_ptr * *tree*, arb_srcptr *weights*, long *len*, long *prec*)

void **_arb_poly_interpolate_fast** (arb_ptr *poly*, arb_srcptr *xs*, arb_srcptr *ys*, long *len*, long *prec*)

void **arb_poly_interpolate_fast** (arb_poly_t *poly*, arb_srcptr *xs*, arb_srcptr *ys*, long *n*, long *prec*)
Recovers the unique polynomial of length at most *n* that interpolates the given *x* and *y* values, using fast Lagrange interpolation. The precomp function takes a precomputed product tree over the *x* values and a vector of interpolation weights as additional inputs.

3.4.14 Differentiation

void **_arb_poly_derivative** (arb_ptr *res*, arb_srcptr *poly*, long *len*, long *prec*)
Sets *{res, len - 1}* to the derivative of *{poly, len}*. Allows aliasing of the input and output.

void **arb_poly_derivative** (arb_poly_t *res*, const arb_poly_t *poly*, long *prec*)
Sets *res* to the derivative of *poly*.

void **_arb_poly_integral** (arb_ptr *res*, arb_srcptr *poly*, long *len*, long *prec*)
Sets *{res, len}* to the integral of *{poly, len - 1}*. Allows aliasing of the input and output.

void **arb_poly_integral** (arb_poly_t *res*, const arb_poly_t *poly*, long *prec*)
Sets *res* to the integral of *poly*.

3.4.15 Transforms

void **_arb_poly_borel_transform** (arb_ptr *res*, arb_srcptr *poly*, long *len*, long *prec*)

void **arb_poly_borel_transform** (arb_poly_t *res*, const arb_poly_t *poly*, long *prec*)
Computes the Borel transform of the input polynomial, mapping $\sum_k a_k x^k$ to $\sum_k (a_k/k!) x^k$. The underscore method allows aliasing.

void **_arb_poly_inv_borel_transform** (arb_ptr *res*, arb_srcptr *poly*, long *len*, long *prec*)

void **arb_poly_inv_borel_transform** (arb_poly_t *res*, const arb_poly_t *poly*, long *prec*)
Computes the inverse Borel transform of the input polynomial, mapping $\sum_k a_k x^k$ to $\sum_k a_k k! x^k$. The underscore method allows aliasing.

void **_arb_poly_binomial_transform_basecase** (arb_ptr *b*, arb_srcptr *a*, long *alen*, long *len*, long *prec*)

void **arb_poly_binomial_transform_basecase** (arb_poly_t *b*, const arb_poly_t *a*, long *len*, long *prec*)

void **_arb_poly_binomial_transform_convolution** (arb_ptr *b*, arb_srcptr *a*, long *alen*, long *len*, long *prec*)

void **arb_poly_binomial_transform_convolution** (arb_poly_t *b*, const arb_poly_t *a*, long *len*, long *prec*)

void **_arb_poly_binomial_transform** (arb_ptr *b*, arb_srcptr *a*, long *alen*, long *len*, long *prec*)

void **arb_poly_binomial_transform** (arb_poly_t *b*, const arb_poly_t *a*, long *len*, long *prec*)
Computes the binomial transform of the input polynomial, truncating the output to length *len*. The binomial transform maps the coefficients a_k in the input polynomial to the coefficients b_k in the output polynomial via $b_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a_k$. The binomial transform is equivalent to the power series composition $f(x) \to (1 - x)^{-1} f(x/(x-1))$, and is its own inverse.

The *basecase* version evaluates coefficients one by one from the definition, generating the binomial coefficients by a recurrence relation.

The *convolution* version uses the identity $T(f(x)) = B^{-1}(e^x B(f(-x)))$ where T denotes the binomial transform operator and B denotes the Borel transform operator. This only costs a single polynomial multiplication, plus some scalar operations.

The default version automatically chooses an algorithm.

The underscore methods do not support aliasing, and assume that the lengths are nonzero.

3.4.16 Powers and elementary functions

void **_arb_poly_pow_ui_trunc_binexp** (arb_ptr *res*, arb_srcptr *f*, long *flen*, ulong *exp*, long *len*, long *prec*)
Sets *{res, len}* to *{f, flen}* raised to the power *exp*, truncated to length *len*. Requires that *len* is no longer than the length of the power as computed without truncation (i.e. no zero-padding is performed). Does not support aliasing of the input and output, and requires that *flen* and *len* are positive. Uses binary expontiation.

void **arb_poly_pow_ui_trunc_binexp** (arb_poly_t *res*, const arb_poly_t *poly*, ulong *exp*, long *len*, long *prec*)
Sets *res* to *poly* raised to the power *exp*, truncated to length *len*. Uses binary exponentiation.

void **_arb_poly_pow_ui** (arb_ptr *res*, arb_srcptr *f*, long *flen*, ulong *exp*, long *prec*)
Sets *res* to *{f, flen}* raised to the power *exp*. Does not support aliasing of the input and output, and requires that *flen* is positive.

void **arb_poly_pow_ui** (arb_poly_t *res*, const arb_poly_t *poly*, ulong *exp*, long *prec*)
Sets *res* to *poly* raised to the power *exp*.

void **_arb_poly_pow_series** (arb_ptr *h*, arb_srcptr *f*, long *flen*, arb_srcptr *g*, long *glen*, long *len*, long *prec*)

Sets *{h, len}* to the power series $f(x)^{g(x)} = \exp(g(x)\log f(x))$ truncated to length *len*. This function detects special cases such as *g* being an exact small integer or $\pm 1/2$, and computes such powers more efficiently. This function does not support aliasing of the output with either of the input operands. It requires that all lengths are positive, and assumes that *flen* and *glen* do not exceed *len*.

void **arb_poly_pow_series** (arb_poly_t *h*, const arb_poly_t *f*, const arb_poly_t *g*, long *len*, long *prec*)

Sets *h* to the power series $f(x)^{g(x)} = \exp(g(x)\log f(x))$ truncated to length *len*. This function detects special cases such as *g* being an exact small integer or $\pm 1/2$, and computes such powers more efficiently.

void **_arb_poly_pow_arb_series** (arb_ptr *h*, arb_srcptr *f*, long *flen*, const arb_t *g*, long *len*, long *prec*)

Sets *{h, len}* to the power series $f(x)^g = \exp(g\log f(x))$ truncated to length *len*. This function detects special cases such as *g* being an exact small integer or $\pm 1/2$, and computes such powers more efficiently. This function does not support aliasing of the output with either of the input operands. It requires that all lengths are positive, and assumes that *flen* does not exceed *len*.

void **arb_poly_pow_arb_series** (arb_poly_t *h*, const arb_poly_t *f*, const arb_t *g*, long *len*, long *prec*)

Sets *h* to the power series $f(x)^g = \exp(g\log f(x))$ truncated to length *len*.

void **_arb_poly_sqrt_series** (arb_ptr *g*, arb_srcptr *h*, long *hlen*, long *n*, long *prec*)

void **arb_poly_sqrt_series** (arb_poly_t *g*, const arb_poly_t *h*, long *n*, long *prec*)

Sets *g* to the power series square root of *h*, truncated to length *n*. Uses division-free Newton iteration for the reciprocal square root, followed by a multiplication.

The underscore method does not support aliasing of the input and output arrays. It requires that *hlen* and *n* are greater than zero.

void **_arb_poly_rsqrt_series** (arb_ptr *g*, arb_srcptr *h*, long *hlen*, long *n*, long *prec*)

void **arb_poly_rsqrt_series** (arb_poly_t *g*, const arb_poly_t *h*, long *n*, long *prec*)

Sets *g* to the reciprocal power series square root of *h*, truncated to length *n*. Uses division-free Newton iteration.

The underscore method does not support aliasing of the input and output arrays. It requires that *hlen* and *n* are greater than zero.

void **_arb_poly_log_series** (arb_ptr *res*, arb_srcptr *f*, long *flen*, long *n*, long *prec*)

void **arb_poly_log_series** (arb_poly_t *res*, const arb_poly_t *f*, long *n*, long *prec*)

Sets *res* to the power series logarithm of *f*, truncated to length *n*. Uses the formula $\log(f(x)) = \int f'(x)/f(x)dx$, adding the logarithm of the constant term in *f* as the constant of integration.

The underscore method supports aliasing of the input and output arrays. It requires that *flen* and *n* are greater than zero.

void **_arb_poly_atan_series** (arb_ptr *res*, arb_srcptr *f*, long *flen*, long *n*, long *prec*)

void **arb_poly_atan_series** (arb_poly_t *res*, const arb_poly_t *f*, long *n*, long *prec*)

void **_arb_poly_asin_series** (arb_ptr *res*, arb_srcptr *f*, long *flen*, long *n*, long *prec*)

void **arb_poly_asin_series** (arb_poly_t *res*, const arb_poly_t *f*, long *n*, long *prec*)

void **_arb_poly_acos_series** (arb_ptr *res*, arb_srcptr *f*, long *flen*, long *n*, long *prec*)

void **arb_poly_acos_series** (arb_poly_t *res*, const arb_poly_t *f*, long *n*, long *prec*)

Sets *res* respectively to the power series inverse tangent, inverse sine and inverse cosine of *f*, truncated to length *n*.

Uses the formulas

$$\tan^{-1}(f(x)) = \int f'(x)/(1+f(x)^2)dx,$$
$$\sin^{-1}(f(x)) = \int f'(x)/(1-f(x)^2)^{1/2}dx,$$
$$\cos^{-1}(f(x)) = -\int f'(x)/(1-f(x)^2)^{1/2}dx,$$

adding the inverse function of the constant term in *f* as the constant of integration.

The underscore methods supports aliasing of the input and output arrays. They require that *flen* and *n* are greater than zero.

void **_arb_poly_exp_series_basecase** (arb_ptr *f*, arb_srcptr *h*, long *hlen*, long *n*, long *prec*)

void **arb_poly_exp_series_basecase** (arb_poly_t *f*, const arb_poly_t *h*, long *n*, long *prec*)

void **_arb_poly_exp_series** (arb_ptr *f*, arb_srcptr *h*, long *hlen*, long *n*, long *prec*)

void **arb_poly_exp_series** (arb_poly_t *f*, const arb_poly_t *h*, long *n*, long *prec*)

Sets f to the power series exponential of h, truncated to length n.

The basecase version uses a simple recurrence for the coefficients, requiring $O(nm)$ operations where m is the length of h.

The main implementation uses Newton iteration, starting from a small number of terms given by the basecase algorithm. The complexity is $O(M(n))$. Redundant operations in the Newton iteration are avoided by using the scheme described in [HZ2004].

The underscore methods support aliasing and allow the input to be shorter than the output, but require the lengths to be nonzero.

void **_arb_poly_sin_cos_series_basecase** (arb_ptr *s*, arb_ptr *c*, arb_srcptr *h*, long *hlen*, long *n*, long *prec*)

void **arb_poly_sin_cos_series_basecase** (arb_poly_t *s*, arb_poly_t *c*, const arb_poly_t *h*, long *n*, long *prec*)

void **_arb_poly_sin_cos_series_tangent** (arb_ptr *s*, arb_ptr *c*, arb_srcptr *h*, long *hlen*, long *n*, long *prec*)

void **arb_poly_sin_cos_series_tangent** (arb_poly_t *s*, arb_poly_t *c*, const arb_poly_t *h*, long *n*, long *prec*)

void **_arb_poly_sin_cos_series** (arb_ptr *s*, arb_ptr *c*, arb_srcptr *h*, long *hlen*, long *n*, long *prec*)

void **arb_poly_sin_cos_series** (arb_poly_t *s*, arb_poly_t *c*, const arb_poly_t *h*, long *n*, long *prec*)

Sets *s* and *c* to the power series sine and cosine of *h*, computed simultaneously.

The *basecase* version uses a simple recurrence for the coefficients, requiring $O(nm)$ operations where m is the length of h.

The *tangent* version uses the tangent half-angle formulas to compute the sine and cosine via `_arb_poly_tan_series()`. This requires $O(M(n))$ operations. When $h = h_0 + h_1$ where the constant term h_0 is nonzero, the evaluation is done as $\sin(h_0+h_1) = \cos(h_0)\sin(h_1) + \sin(h_0)\cos(h_1)$, $\cos(h_0+h_1) = \cos(h_0)\cos(h_1) - \sin(h_0)\sin(h_1)$, to improve accuracy and avoid dividing by zero at the poles of the tangent function.

The default version automatically selects between the *basecase* and *tangent* algorithms depending on the input.

The underscore methods support aliasing and require the lengths to be nonzero.

void **_arb_poly_sin_series** (arb_ptr *s*, arb_srcptr *h*, long *hlen*, long *n*, long *prec*) void **arb_poly_sin_series** (arb_poly_t *s*, const arb_poly_t *h*, long *n*, long *prec*)

void **_arb_poly_cos_series** (arb_ptr *c*, arb_srcptr *h*, long *hlen*, long *n*, long *prec*)

void **arb_poly_cos_series** (arb_poly_t *c*, const arb_poly_t *h*, long *n*, long *prec*)

Respectively evaluates the power series sine or cosine. These functions simply wrap `_arb_poly_sin_cos_series()`. The underscore methods support aliasing and require the lengths to be nonzero.

void **_arb_poly_tan_series** (arb_ptr *g*, arb_srcptr *h*, long *hlen*, long *len*, long *prec*)

void **arb_poly_tan_series** (arb_poly_t *g*, const arb_poly_t *h*, long *n*, long *prec*)

Sets *g* to the power series tangent of *h*.

For small *n* takes the quotient of the sine and cosine as computed using the basecase algorithm. For large *n*, uses Newton iteration to invert the inverse tangent series. The complexity is $O(M(n))$.

The underscore version does not support aliasing, and requires the lengths to be nonzero.

3.4.17 Gamma function and factorials

void **_arb_poly_gamma_series** (arb_ptr *res*, arb_srcptr *h*, long *hlen*, long *n*, long *prec*)

void **arb_poly_gamma_series** (arb_poly_t *res*, const arb_poly_t *h*, long *n*, long *prec*)

void **_arb_poly_rgamma_series** (arb_ptr *res*, arb_srcptr *h*, long *hlen*, long *n*, long *prec*)

void **arb_poly_rgamma_series** (arb_poly_t *res*, const arb_poly_t *h*, long *n*, long *prec*)

void **_arb_poly_lgamma_series** (arb_ptr *res*, arb_srcptr *h*, long *hlen*, long *n*, long *prec*)

void **arb_poly_lgamma_series** (arb_poly_t *res*, const arb_poly_t *h*, long *n*, long *prec*)

Sets *res* to the series expansion of $\Gamma(h(x))$, $1/\Gamma(h(x))$, or $\log \Gamma(h(x))$, truncated to length *n*.

These functions first generate the Taylor series at the constant term of *h*, and then call `_arb_poly_compose_series()`. The Taylor coefficients are generated using the Riemann zeta function if the constant term of *h* is a small integer, and with Stirling's series otherwise.

The underscore methods support aliasing of the input and output arrays, and require that *hlen* and *n* are greater than zero.

void **_arb_poly_rising_ui_series** (arb_ptr *res*, arb_srcptr *f*, long *flen*, ulong *r*, long *trunc*, long *prec*)

void **arb_poly_rising_ui_series** (arb_poly_t *res*, const arb_poly_t *f*, ulong *r*, long *trunc*, long *prec*)

Sets *res* to the rising factorial $(f)(f+1)(f+2)\cdots(f+r-1)$, truncated to length *trunc*. The underscore method assumes that *flen*, *r* and *trunc* are at least 1, and does not support aliasing. Uses binary splitting.

3.4.18 Zeta function

void **arb_poly_zeta_series** (arb_poly_t *res*, const arb_poly_t *s*, const arb_t *a*, int *deflate*, long *n*, long *prec*)

Sets *res* to the Hurwitz zeta function $\zeta(s,a)$ where s a power series and a is a constant, truncated to length *n*. To evaluate the usual Riemann zeta function, set $a = 1$.

If *deflate* is nonzero, evaluates $\zeta(s,a) + 1/(1-s)$, which is well-defined as a limit when the constant term of s is 1. In particular, expanding $\zeta(s,a) + 1/(1-s)$ with $s = 1 + x$ gives the Stieltjes constants

$$\sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \gamma_k(a) x^k.$$

If $a = 1$, this implementation uses the reflection formula if the midpoint of the constant term of s is negative.

void **_arb_poly_riemann_siegel_theta_series**(arb_ptr *res*, arb_srcptr *h*, long *hlen*, long *n*, long *prec*)

void **arb_poly_riemann_siegel_theta_series**(arb_poly_t *res*, const arb_poly_t *h*, long *n*, long *prec*)

Sets *res* to the series expansion of the Riemann-Siegel theta function

$$\theta(h) = \arg\left(\Gamma\left(\frac{2ih+1}{4}\right)\right) - \frac{\log \pi}{2}h$$

where the argument of the gamma function is chosen continuously as the imaginary part of the log gamma function.

The underscore method does not support aliasing of the input and output arrays, and requires that the lengths are greater than zero.

void **_arb_poly_riemann_siegel_z_series**(arb_ptr *res*, arb_srcptr *h*, long *hlen*, long *n*, long *prec*)

void **arb_poly_riemann_siegel_z_series**(arb_poly_t *res*, const arb_poly_t *h*, long *n*, long *prec*)

Sets *res* to the series expansion of the Riemann-Siegel Z-function

$$Z(h) = e^{i\theta(h)}\zeta(1/2 + ih).$$

The zeros of the Z-function on the real line precisely correspond to the imaginary parts of the zeros of the Riemann zeta function on the critical line.

The underscore method supports aliasing of the input and output arrays, and requires that the lengths are greater than zero.

3.4.19 Root-finding

void **_arb_poly_newton_convergence_factor**(arf_t *convergence_factor*, arb_srcptr *poly*, long *len*, const arb_t *convergence_interval*, long *prec*)

Given an interval I specified by *convergence_interval*, evaluates a bound for $C = \sup_{t,u \in I} \frac{1}{2}|f''(t)|/|f'(u)|$, where f is the polynomial defined by the coefficients *{poly, len}*. The bound is obtained by evaluating $f'(I)$ and $f''(I)$ directly. If f has large coefficients, I must be extremely precise in order to get a finite factor.

int **_arb_poly_newton_step**(arb_t *xnew*, arb_srcptr *poly*, long *len*, const arb_t *x*, const arb_t *convergence_interval*, const arf_t *convergence_factor*, long *prec*)

Performs a single step with Newton's method.

The input consists of the polynomial f specified by the coefficients *{poly, len}*, an interval $x = [m - r, m + r]$ known to contain a single root of f, an interval I (*convergence_interval*) containing x with an associated bound (*convergence_factor*) for $C = \sup_{t,u \in I} \frac{1}{2}|f''(t)|/|f'(u)|$, and a working precision *prec*.

The Newton update consists of setting $x' = [m' - r', m' + r']$ where $m' = m - f(m)/f'(m)$ and $r' = Cr^2$. The expression $m - f(m)/f'(m)$ is evaluated using ball arithmetic at a working precision of *prec* bits, and the rounding error during this evaluation is accounted for in the output. We now check that $x' \in I$ and $m' < m$. If both conditions are satisfied, we set *xnew* to x' and return nonzero. If either condition fails, we set *xnew* to x and return zero, indicating that no progress was made.

void **_arb_poly_newton_refine_root**(arb_t *r*, arb_srcptr *poly*, long *len*, const arb_t *start*, const arb_t *convergence_interval*, const arf_t *convergence_factor*, long *eval_extra_prec*, long *prec*)

Refines a precise estimate of a polynomial root to high precision by performing several Newton steps, using nearly optimally chosen doubling precision steps.

The inputs are defined as for *_arb_poly_newton_step*, except for the precision parameters: *prec* is the target accuracy and *eval_extra_prec* is the estimated number of guard bits that need to be added to evaluate the polynomial accurately close to the root (typically, if the polynomial has large coefficients of alternating signs, this needs to be approximately the bit size of the coefficients).

3.4.20 Other special polynomials

void **_arb_poly_swinnerton_dyer_ui** (arb_ptr *poly*, ulong *n*, long *trunc*, long *prec*)

void **arb_poly_swinnerton_dyer_ui** (arb_poly_t *poly*, ulong *n*, long *prec*)
Computes the Swinnerton-Dyer polynomial S_n, which has degree 2^n and is the rational minimal polynomial of the sum of the square roots of the first *n* prime numbers.

If *prec* is set to zero, a precision is chosen automatically such that `arb_poly_get_unique_fmpz_poly()` should be successful. Otherwise a working precision of *prec* bits is used.

The underscore version accepts an additional *trunc* parameter. Even when computing a truncated polynomial, the array *poly* must have room for $2^n + 1$ coefficients, used as temporary space.

3.5 arb_mat.h – matrices over the real numbers

An `arb_mat_t` represents a dense matrix over the real numbers, implemented as an array of entries of type `arb_struct`.

The dimension (number of rows and columns) of a matrix is fixed at initialization, and the user must ensure that inputs and outputs to an operation have compatible dimensions. The number of rows or columns in a matrix can be zero.

3.5.1 Types, macros and constants

arb_mat_struct

arb_mat_t
Contains a pointer to a flat array of the entries (entries), an array of pointers to the start of each row (rows), and the number of rows (r) and columns (c).

An *arb_mat_t* is defined as an array of length one of type *arb_mat_struct*, permitting an *arb_mat_t* to be passed by reference.

arb_mat_entry (mat, i, j)
Macro giving a pointer to the entry at row *i* and column *j*.

arb_mat_nrows (mat)
Returns the number of rows of the matrix.

arb_mat_ncols (mat)
Returns the number of columns of the matrix.

3.5.2 Memory management

void **arb_mat_init** (arb_mat_t *mat*, long *r*, long *c*)
Initializes the matrix, setting it to the zero matrix with *r* rows and *c* columns.

void **arb_mat_clear** (arb_mat_t *mat*)
Clears the matrix, deallocating all entries.

3.5.3 Conversions

void **arb_mat_set** (arb_mat_t *dest*, const arb_mat_t *src*)

void **arb_mat_set_fmpz_mat** (arb_mat_t *dest*, const fmpz_mat_t *src*)

void **arb_mat_set_fmpq_mat** (arb_mat_t *dest*, const fmpq_mat_t *src*, long *prec*)
Sets *dest* to *src*. The operands must have identical dimensions.

3.5.4 Random generation

void **arb_mat_randtest** (arb_mat_t *mat*, flint_rand_t *state*, long *prec*, long *mag_bits*)
Sets *mat* to a random matrix with up to *prec* bits of precision and with exponents of width up to *mag_bits*.

3.5.5 Input and output

void **arb_mat_printd** (const arb_mat_t *mat*, long *digits*)
Prints each entry in the matrix with the specified number of decimal digits.

3.5.6 Comparisons

int **arb_mat_equal** (const arb_mat_t *mat1*, const arb_mat_t *mat2*)
Returns nonzero iff the matrices have the same dimensions and identical entries.

int **arb_mat_overlaps** (const arb_mat_t *mat1*, const arb_mat_t *mat2*)
Returns nonzero iff the matrices have the same dimensions and each entry in *mat1* overlaps with the corresponding entry in *mat2*.

int **arb_mat_contains** (const arb_mat_t *mat1*, const arb_mat_t *mat2*)

int **arb_mat_contains_fmpz_mat** (const arb_mat_t *mat1*, const fmpz_mat_t *mat2*)

int **arb_mat_contains_fmpq_mat** (const arb_mat_t *mat1*, const fmpq_mat_t *mat2*)
Returns nonzero iff the matrices have the same dimensions and each entry in *mat2* is contained in the corresponding entry in *mat1*.

3.5.7 Special matrices

void **arb_mat_zero** (arb_mat_t *mat*)
Sets all entries in mat to zero.

void **arb_mat_one** (arb_mat_t *mat*)
Sets the entries on the main diagonal to ones, and all other entries to zero.

3.5.8 Norms

void **arb_mat_bound_inf_norm** (mag_t *b*, const arb_mat_t *A*)
Sets *b* to an upper bound for the infinity norm (i.e. the largest absolute value row sum) of *A*.

3.5.9 Arithmetic

void **arb_mat_neg** (arb_mat_t *dest*, const arb_mat_t *src*)
Sets *dest* to the exact negation of *src*. The operands must have the same dimensions.

void **arb_mat_add** (arb_mat_t *res*, const arb_mat_t *mat1*, const arb_mat_t *mat2*, long *prec*)
Sets res to the sum of *mat1* and *mat2*. The operands must have the same dimensions.

void **arb_mat_sub** (arb_mat_t *res*, const arb_mat_t *mat1*, const arb_mat_t *mat2*, long *prec*)
Sets *res* to the difference of *mat1* and *mat2*. The operands must have the same dimensions.

void **arb_mat_mul_classical** (arb_mat_t *C*, const arb_mat_t *A*, const arb_mat_t *B*, long *prec*)

void **arb_mat_mul_threaded** (arb_mat_t *C*, const arb_mat_t *A*, const arb_mat_t *B*, long *prec*)

void **arb_mat_mul** (arb_mat_t *res*, const arb_mat_t *mat1*, const arb_mat_t *mat2*, long *prec*)
Sets *res* to the matrix product of *mat1* and *mat2*. The operands must have compatible dimensions for matrix multiplication.

The *threaded* version splits the computation over the number of threads returned by *flint_get_num_threads()*. The default version automatically calls the *threaded* version if the matrices are sufficiently large and more than one thread can be used.

void **arb_mat_pow_ui** (arb_mat_t *res*, const arb_mat_t *mat*, ulong *exp*, long *prec*)
Sets *res* to *mat* raised to the power *exp*. Requires that *mat* is a square matrix.

3.5.10 Scalar arithmetic

void **arb_mat_scalar_mul_2exp_si** (arb_mat_t *B*, const arb_mat_t *A*, long *c*)
Sets *B* to *A* multiplied by 2^c.

void **arb_mat_scalar_addmul_si** (arb_mat_t *B*, const arb_mat_t *A*, long *c*, long *prec*)

void **arb_mat_scalar_addmul_fmpz** (arb_mat_t *B*, const arb_mat_t *A*, const fmpz_t *c*, long *prec*)

void **arb_mat_scalar_addmul_arb** (arb_mat_t *B*, const arb_mat_t *A*, const arb_t *c*, long *prec*)
Sets *B* to $B + A \times c$.

void **arb_mat_scalar_mul_si** (arb_mat_t *B*, const arb_mat_t *A*, long *c*, long *prec*)

void **arb_mat_scalar_mul_fmpz** (arb_mat_t *B*, const arb_mat_t *A*, const fmpz_t *c*, long *prec*)

void **arb_mat_scalar_mul_arb** (arb_mat_t *B*, const arb_mat_t *A*, const arb_t *c*, long *prec*)
Sets *B* to $A \times c$.

void **arb_mat_scalar_div_si** (arb_mat_t *B*, const arb_mat_t *A*, long *c*, long *prec*)

void **arb_mat_scalar_div_fmpz** (arb_mat_t *B*, const arb_mat_t *A*, const fmpz_t *c*, long *prec*)

void **arb_mat_scalar_div_arb** (arb_mat_t *B*, const arb_mat_t *A*, const arb_t *c*, long *prec*)
Sets *B* to A/c.

3.5.11 Gaussian elimination and solving

int **arb_mat_lu** (long * *perm*, arb_mat_t *LU*, const arb_mat_t *A*, long *prec*)
Given an $n \times n$ matrix A, computes an LU decomposition $PLU = A$ using Gaussian elimination with partial pivoting. The input and output matrices can be the same, performing the decomposition in-place.

Entry i in the permutation vector perm is set to the row index in the input matrix corresponding to row i in the output matrix.

The algorithm succeeds and returns nonzero if it can find n invertible (i.e. not containing zero) pivot entries. This guarantees that the matrix is invertible.

The algorithm fails and returns zero, leaving the entries in P and LU undefined, if it cannot find n invertible pivot elements. In this case, either the matrix is singular, the input matrix was computed to insufficient precision, or the LU decomposition was attempted at insufficient precision.

void **arb_mat_solve_lu_precomp** (arb_mat_t *X*, const long * *perm*, const arb_mat_t *LU*, const arb_mat_t *B*, long *prec*)
Solves $AX = B$ given the precomputed nonsingular LU decomposition $A = PLU$. The matrices X and B are allowed to be aliased with each other, but X is not allowed to be aliased with LU.

int **arb_mat_solve** (arb_mat_t *X*, const arb_mat_t *A*, const arb_mat_t *B*, long *prec*)
Solves $AX = B$ where A is a nonsingular $n \times n$ matrix and X and B are $n \times m$ matrices, using LU decomposition.

If $m > 0$ and A cannot be inverted numerically (indicating either that A is singular or that the precision is insufficient), the values in the output matrix are left undefined and zero is returned. A nonzero return value guarantees that A is invertible and that the exact solution matrix is contained in the output.

int **arb_mat_inv** (arb_mat_t *X*, const arb_mat_t *A*, long *prec*)
Sets $X = A^{-1}$ where A is a square matrix, computed by solving the system $AX = I$.

If A cannot be inverted numerically (indicating either that A is singular or that the precision is insufficient), the values in the output matrix are left undefined and zero is returned. A nonzero return value guarantees that the matrix is invertible and that the exact inverse is contained in the output.

void **arb_mat_det** (arb_t *det*, const arb_mat_t *A*, long *prec*)
Computes the determinant of the matrix, using Gaussian elimination with partial pivoting. If at some point an invertible pivot element cannot be found, the elimination is stopped and the magnitude of the determinant of the remaining submatrix is bounded using Hadamard's inequality.

3.5.12 Characteristic polynomial

void **_arb_mat_charpoly** (arb_ptr *cp*, const arb_mat_t *mat*, long *prec*)

void **arb_mat_charpoly** (arb_poly_t *cp*, const arb_mat_t *mat*, long *prec*)
Sets *cp* to the characteristic polynomial of *mat* which must be a square matrix. If the matrix has *n* rows, the underscore method requires space for $n + 1$ output coefficients. Employs a division-free algorithm using $O(n^4)$ operations.

3.5.13 Special functions

void **arb_mat_exp** (arb_mat_t *B*, const arb_mat_t *A*, long *prec*)
Sets *B* to the exponential of the matrix *A*, defined by the Taylor series

$$\exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

The function is evaluated as $\exp(A/2^r)^{2^r}$, where r is chosen to give rapid convergence of the Taylor series. The series is evaluated using rectangular splitting. If $\|A/2^r\| \le c$ and $N \ge 2c$, we bound the entrywise error when truncating the Taylor series before term N by $2c^N/N!$.

3.6 arb_calc.h – calculus with real-valued functions

This module provides functions for operations of calculus over the real numbers (intended to include root-finding, optimization, integration, and so on). It is planned that the module will include two types of algorithms:

- Interval algorithms that give provably correct results. An example would be numerical integration on an interval by dividing the interval into small balls and evaluating the function on each ball, giving rigorous upper and lower bounds.

- Conventional numerical algorithms that use heuristics to estimate the accuracy of a result, without guaranteeing that it is correct. An example would be numerical integration based on pointwise evaluation, where the error is estimated by comparing the results with two different sets of evaluation points. Ball arithmetic then still tracks the accuracy of the function evaluations.

Any algorithms of the second kind will be clearly marked as such.

3.6.1 Types, macros and constants

arb_calc_func_t
Typedef for a pointer to a function with signature:

```
int func(arb_ptr out, const arb_t inp, void * param, long order, long prec)
```

implementing a univariate real function $f(x)$. When called, *func* should write to *out* the first *order* coefficients in the Taylor series expansion of $f(x)$ at the point *inp*, evaluated at a precision of *prec* bits. The *param* argument may be used to pass through additional parameters to the function. The return value is reserved for future use as an error code. It can be assumed that *out* and *inp* are not aliased and that *order* is positive.

ARB_CALC_SUCCESS
Return value indicating that an operation is successful.

ARB_CALC_IMPRECISE_INPUT
Return value indicating that the input to a function probably needs to be computed more accurately.

ARB_CALC_NO_CONVERGENCE
Return value indicating that an algorithm has failed to convergence, possibly due to the problem not having a solution, the algorithm not being applicable, or the precision being insufficient

3.6.2 Debugging

int **arb_calc_verbose**
If set, enables printing information about the calculation to standard output.

3.6.3 Subdivision-based root finding

arf_interval_struct

arf_interval_interval_t
An `arf_interval_struct` consists of a pair of `arf_struct`, representing an interval used for subdivision-based root-finding. An `arf_interval_t` is defined as an array of length one of type `arf_interval_struct`, permitting an `arf_interval_t` to be passed by reference.

arf_interval_ptr
Alias for `arf_interval_struct *`, used for vectors of intervals.

arf_interval_srcptr
Alias for `const arf_interval_struct *`, used for vectors of intervals.

void **arf_interval_init** (arf_interval_t *v*)

void **arf_interval_clear** (arf_interval_t *v*)

arf_interval_ptr **_arf_interval_vec_init** (long *n*)

void **_arf_interval_vec_clear** (arf_interval_ptr *v*, long *n*)

void **arf_interval_set** (arf_interval_t *v*, const arf_interval_t *u*) void **arf_interval_swap** (arf_interval_t *v*, arf_interval_t *u*)

void **arf_interval_get_arb** (arb_t *x*, const arf_interval_t *v*, long *prec*)

void **arf_interval_printd** (const arf_interval_t *v*, long *n*)

Helper functions for endpoint-based intervals.

long **arb_calc_isolate_roots** (arf_interval_ptr * *found*, int ** *flags*, arb_calc_func_t *func*, void * *param*, const arf_interval_t *interval*, long *maxdepth*, long *maxeval*, long *maxfound*, long *prec*)

Rigorously isolates single roots of a real analytic function on the interior of an interval.

This routine writes an array of *n* interesting subintervals of *interval* to *found* and corresponding flags to *flags*, returning the integer *n*. The output has the following properties:

- The function has no roots on *interval* outside of the output subintervals.
- Subintervals are sorted in increasing order (with no overlap except possibly starting and ending with the same point).
- Subintervals with a flag of 1 contain exactly one (single) root.
- Subintervals with any other flag may or may not contain roots.

If no flags other than 1 occur, all roots of the function on *interval* have been isolated. If there are output subintervals on which the existence or nonexistence of roots could not be determined, the user may attempt further searches on those subintervals (possibly with increased precision and/or increased bounds for the breaking criteria). Note that roots of multiplicity higher than one and roots located exactly at endpoints cannot be isolated by the algorithm.

The following breaking criteria are implemented:

- At most *maxdepth* recursive subdivisions are attempted. The smallest details that can be distinguished are therefore about $2^{-\mathrm{maxdepth}}$ times the width of *interval*. A typical, reasonable value might be between 20 and 50.
- If the total number of tested subintervals exceeds *maxeval*, the algorithm is terminated and any untested subintervals are added to the output. The total number of calls to *func* is thereby restricted to a small multiple of *maxeval* (the actual count can be slightly higher depending on implementation details). A typical, reasonable value might be between 100 and 100000.
- The algorithm terminates if *maxfound* roots have been isolated. In particular, setting *maxfound* to 1 can be used to locate just one root of the function even if there are numerous roots. To try to find all roots, *LONG_MAX* may be passed.

The argument *prec* denotes the precision used to evaluate the function. It is possibly also used for some other arithmetic operations performed internally by the algorithm. Note that it probably does not make sense for *maxdepth* to exceed *prec*.

Warning: it is assumed that subdivision points of *interval* can be represented exactly as floating-point numbers in memory. Do not pass $1 \pm 2^{-10^{100}}$ as input.

int **arb_calc_refine_root_bisect** (arf_interval_t *r*, arb_calc_func_t *func*, void * *param*, const arf_interval_t *start*, long *iter*, long *prec*)

Given an interval *start* known to contain a single root of *func*, refines it using *iter* bisection steps. The algorithm can return a failure code if the sign of the function at an evaluation point is ambiguous. The output *r* is set to a valid isolating interval (possibly just *start*) even if the algorithm fails.

3.6.4 Newton-based root finding

void **arb_calc_newton_conv_factor** (arf_t *conv_factor*, arb_calc_func_t *func*, void * *param*, const arb_t *conv_region*, long *prec*)

Given an interval I specified by *conv_region*, evaluates a bound for $C = \sup_{t,u \in I} \frac{1}{2}|f''(t)|/|f'(u)|$, where f is the function specified by *func* and *param*. The bound is obtained by evaluating $f'(I)$ and $f''(I)$ directly. If f is ill-conditioned, I may need to be extremely precise in order to get an effective, finite bound for *C*.

int **arb_calc_newton_step** (arb_t *xnew*, arb_calc_func_t *func*, void * *param*, const arb_t *x*, const arb_t *conv_region*, const arf_t *conv_factor*, long *prec*)

Performs a single step with an interval version of Newton's method. The input consists of the function f specified by *func* and *param*, a ball $x = [m - r, m + r]$ known to contain a single root of f, a ball I (*conv_region*) containing x with an associated bound (*conv_factor*) for $C = \sup_{t,u \in I} \frac{1}{2}|f''(t)|/|f'(u)|$, and a working precision *prec*.

The Newton update consists of setting $x' = [m' - r', m' + r']$ where $m' = m - f(m)/f'(m)$ and $r' = Cr^2$. The expression $m - f(m)/f'(m)$ is evaluated using ball arithmetic at a working precision of *prec* bits, and the rounding error during this evaluation is accounted for in the output. We now check that $x' \in I$ and $r' < r$. If both conditions are satisfied, we set *xnew* to x' and return *ARB_CALC_SUCCESS*. If either condition fails, we set *xnew* to x and return *ARB_CALC_NO_CONVERGENCE*, indicating that no progress is made.

int **arb_calc_refine_root_newton** (arb_t *r*, arb_calc_func_t *func*, void * *param*, const arb_t *start*, const arb_t *conv_region*, const arf_t *conv_factor*, long *eval_extra_prec*, long *prec*)

Refines a precise estimate of a single root of a function to high precision by performing several Newton steps, using nearly optimally chosen doubling precision steps.

The inputs are defined as for *arb_calc_newton_step*, except for the precision parameters: *prec* is the target accuracy and *eval_extra_prec* is the estimated number of guard bits that need to be added to evaluate the function accurately close to the root (for example, if the function is a polynomial with large coefficients of alternating signs and Horner's rule is used to evaluate it, the extra precision should typically be approximately the bit size of the coefficients).

This function returns *ARB_CALC_SUCCESS* if all attempted Newton steps are successful (note that this does not guarantee that the computed root is accurate to *prec* bits, which has to be verified by the user), only that it is more accurate than the starting ball.

On failure, *ARB_CALC_IMPRECISE_INPUT* or *ARB_CALC_NO_CONVERGENCE* may be returned. In this case, *r* is set to a ball for the root which is valid but likely does have full accuracy (it can possibly just be equal to the starting ball).

3.7 acb.h – complex numbers

An `acb_t` represents a complex number with error bounds. An `acb_t` consists of a pair of real number balls of type `arb_struct`, representing the real and imaginary part with separate error bounds.

An `acb_t` thus represents a rectangle $[m_1 - r_1, m_1 + r_1] + [m_2 - r_2, m_2 + r_2]i$ in the complex plane. This is used instead of a disk or square representation (consisting of a complex floating-point midpoint with a single radius), since it allows implementing many operations more conveniently by splitting into ball operations on the real and imaginary parts. It also allows tracking when complex numbers have an exact (for example exactly zero) real part and an inexact imaginary part, or vice versa.

The interface for the `acb_t` type is slightly less developed than that for the `arb_t` type. In many cases, the user can easily perform missing operations by directly manipulating the real and imaginary parts.

3.7.1 Types, macros and constants

acb_struct

acb_t
An *acb_struct* consists of a pair of *arb_struct*:s. An *acb_t* is defined as an array of length one of type *acb_struct*, permitting an *acb_t* to be passed by reference.

acb_ptr
Alias for `acb_struct *`, used for vectors of numbers.

acb_srcptr
Alias for `const acb_struct *`, used for vectors of numbers when passed as constant input to functions.

acb_realref(x)
Macro returning a pointer to the real part of x as an *arb_t*.

arb_imagref(x)
Macro returning a pointer to the imaginary part of x as an *arb_t*.

3.7.2 Memory management

void **acb_init**(arb_t x)
Initializes the variable x for use, and sets its value to zero.

void **acb_clear**(acb_t x)
Clears the variable x, freeing or recycling its allocated memory.

acb_ptr **_acb_vec_init**(long n)
Returns a pointer to an array of n initialized *acb_struct*:s.

void **_acb_vec_clear**(acb_ptr v, long n)
Clears an array of n initialized *acb_struct*:s.

3.7.3 Basic manipulation

int **acb_is_zero**(const acb_t z)
Returns nonzero iff z is zero.

int **acb_is_one**(const acb_t z)
Returns nonzero iff z is exactly 1.

int **acb_is_exact**(const acb_t z)
Returns nonzero iff z is exact.

int **acb_is_int**(const acb_t z)
Returns nonzero iff z is an exact integer.

void **acb_zero**(acb_t z)

void **acb_one**(acb_t z)

void **acb_onei**(acb_t z)
Sets z respectively to 0, 1, $i = \sqrt{-1}$.

void **acb_set**(acb_t z, const acb_t x)

void **acb_set_ui**(acb_t z, long x)

void **acb_set_si**(acb_t z, long x)

void **acb_set_fmpz** (acb_t *z*, const fmpz_t *x*)

void **acb_set_arb** (acb_t *z*, const arb_t *c*)
Sets *z* to the value of *x*.

void **acb_set_fmpq** (acb_t *z*, const fmpq_t *x*, long *prec*)

void **acb_set_round** (acb_t *z*, const acb_t *x*, long *prec*)

void **acb_set_round_fmpz** (acb_t *z*, const fmpz_t *x*, long *prec*)

void **acb_set_round_arb** (acb_t *z*, const arb_t *x*, long *prec*)
Sets *z* to *x*, rounded to *prec* bits.

void **acb_swap** (acb_t *z*, acb_t *x*)
Swaps *z* and *x* efficiently.

3.7.4 Input and output

void **acb_print** (const acb_t *x*)
Prints the internal representation of *x*.

void **acb_printd** (const acb_t *z*, long *digits*)
Prints *x* in decimal. The printed value of the radius is not adjusted to compensate for the fact that the binary-to-decimal conversion of both the midpoint and the radius introduces additional error.

3.7.5 Random number generation

void **acb_randtest** (acb_t *z*, flint_rand_t *state*, long *prec*, long *mag_bits*)
Generates a random complex number by generating separate random real and imaginary parts.

void **acb_randtest_special** (acb_t *z*, flint_rand_t *state*, long *prec*, long *mag_bits*)
Generates a random complex number by generating separate random real and imaginary parts. Also generates NaNs and infinities.

void **acb_randtest_precise** (acb_t *z*, flint_rand_t *state*, long *prec*, long *mag_bits*)
Generates a random complex number with precise real and imaginary parts.

3.7.6 Precision and comparisons

int **acb_equal** (const acb_t *x*, const acb_t *y*)
Returns nonzero iff *x* and *y* are identical as sets, i.e. if the real and imaginary parts are equal as balls.

Note that this is not the same thing as testing whether both *x* and *y* certainly represent the same complex number, unless either *x* or *y* is exact (and neither contains NaN). To test whether both operands *might* represent the same mathematical quantity, use `acb_overlaps()` or `acb_contains()`, depending on the circumstance.

int **acb_overlaps** (const acb_t *x*, const acb_t *y*)
Returns nonzero iff *x* and *y* have some point in common.

void **acb_get_abs_ubound_arf** (arf_t *u*, const acb_t *z*, long *prec*)
Sets *u* to an upper bound for the absolute value of *z*, computed using a working precision of *prec* bits.

void **acb_get_abs_lbound_arf** (arf_t *u*, const acb_t *z*, long *prec*)
Sets *u* to a lower bound for the absolute value of *z*, computed using a working precision of *prec* bits.

void **acb_get_rad_ubound_arf** (arf_t *u*, const acb_t *z*, long *prec*)
Sets *u* to an upper bound for the error radius of *z* (the value is currently not computed tightly).

void **acb_get_mag** (mag_t *u*, const acb_t *x*)
Sets *u* to an upper bound for the absolute value of *x*.

void **acb_get_mag_lower** (mag_t *u*, const acb_t *x*)
Sets *u* to a lower bound for the absolute value of *x*.

int **acb_contains_fmpq** (const acb_t *x*, const fmpq_t *y*)

int **acb_contains_fmpz** (const acb_t *x*, const fmpz_t *y*)

int **acb_contains** (const acb_t *x*, const acb_t *y*)
Returns nonzero iff *y* is contained in *x*.

int **acb_contains_zero** (const acb_t *x*)
Returns nonzero iff zero is contained in *x*.

long **acb_bits** (const acb_t *x*)
Returns the maximum of *arb_bits* applied to the real and imaginary parts of *x*, i.e. the minimum precision sufficient to represent *x* exactly.

void **acb_trim** (acb_t *y*, const acb_t *x*)
Sets *y* to a a copy of *x* with both the real and imaginary parts trimmed (see `arb_trim()`).

int **acb_is_real** (const acb_t *x*)
Returns nonzero iff the imaginary part of *x* is zero. It does not test whether the real part of *x* also is finite.

int **acb_get_unique_fmpz** (fmpz_t *z*, const acb_t *x*)
If *x* contains a unique integer, sets *z* to that value and returns nonzero. Otherwise (if *x* represents no integers or more than one integer), returns zero.

3.7.7 Complex parts

void **acb_arg** (arb_t *r*, const acb_t *z*, long *prec*)
Sets *r* to a real interval containing the complex argument (phase) of *z*. We define the complex argument have a discontinuity on $(-\infty, 0]$, with the special value $\arg(0) = 0$, and $\arg(a + 0i) = \pi$ for $a < 0$. Equivalently, if $z = a + bi$, the argument is given by $\operatorname{atan2}(b, a)$ (see `arb_atan2()`).

void **acb_abs** (arb_t *r*, const acb_t *z*, long *prec*)
Sets *r* to the absolute value of *z*.

3.7.8 Arithmetic

void **acb_neg** (acb_t *z*, const acb_t *x*)
Sets *z* to the negation of *x*.

void **acb_conj** (acb_t *z*, const acb_t *x*)
Sets *z* to the complex conjugate of *x*.

void **acb_add_ui** (acb_t *z*, const acb_t *x*, ulong *y*, long *prec*)

void **acb_add_fmpz** (acb_t *z*, const acb_t *x*, const fmpz_t *y*, long *prec*)

void **acb_add_arb** (acb_t *z*, const acb_t *x*, const arb_t *y*, long *prec*)

void **acb_add** (acb_t *z*, const acb_t *x*, const acb_t *y*, long *prec*)
Sets *z* to the sum of *x* and *y*.

void **acb_sub_ui** (acb_t *z*, const acb_t *x*, ulong *y*, long *prec*)

void **acb_sub_fmpz** (acb_t *z*, const acb_t *x*, const fmpz_t *y*, long *prec*) void **acb_sub_arb** (acb_t *z*, const acb_t *x*, const arb_t *y*, long *prec*)

void **acb_sub** (acb_t *z*, const acb_t *x*, const acb_t *y*, long *prec*)
Sets z to the difference of x and y.

void **acb_mul_onei** (acb_t *z*, const acb_t *x*)
Sets z to x multiplied by the imaginary unit.

void **acb_mul_ui** (acb_t *z*, const acb_t *x*, ulong *y*, long *prec*)

void **acb_mul_si** (acb_t *z*, const acb_t *x*, long *y*, long *prec*)

void **acb_mul_fmpz** (acb_t *z*, const acb_t *x*, const fmpz_t *y*, long *prec*)

void **acb_mul_arb** (acb_t *z*, const acb_t *x*, const arb_t *y*, long *prec*)
Sets z to the product of x and y.

void **acb_mul** (acb_t *z*, const acb_t *x*, const acb_t *y*, long *prec*)
Sets z to the product of x and y. If at least one part of x or y is zero, the operations is reduced to two real multiplications. If x and y are the same pointers, they are assumed to represent the same mathematical quantity and the squaring formula is used.

void **acb_mul_2exp_si** (acb_t *z*, const acb_t *x*, long *e*)

void **acb_mul_2exp_fmpz** (acb_t *z*, const acb_t *x*, const fmpz_t *e*)
Sets z to x multiplied by 2^e, without rounding.

void **acb_cube** (acb_t *z*, const acb_t *x*, long *prec*)
Sets z to x cubed, computed efficiently using two real squarings, two real multiplications, and scalar operations.

void **acb_addmul** (acb_t *z*, const acb_t *x*, const acb_t *y*, long *prec*)

void **acb_addmul_ui** (acb_t *z*, const acb_t *x*, ulong *y*, long *prec*)

void **acb_addmul_si** (acb_t *z*, const acb_t *x*, long *y*, long *prec*)

void **acb_addmul_fmpz** (acb_t *z*, const acb_t *x*, const fmpz_t *y*, long *prec*)

void **acb_addmul_arb** (acb_t *z*, const acb_t *x*, const arb_t *y*, long *prec*)
Sets z to z plus the product of x and y.

void **acb_submul** (acb_t *z*, const acb_t *x*, const acb_t *y*, long *prec*)

void **acb_submul_ui** (acb_t *z*, const acb_t *x*, ulong *y*, long *prec*)

void **acb_submul_si** (acb_t *z*, const acb_t *x*, long *y*, long *prec*)

void **acb_submul_fmpz** (acb_t *z*, const acb_t *x*, const fmpz_t *y*, long *prec*)

void **acb_submul_arb** (acb_t *z*, const acb_t *x*, const arb_t *y*, long *prec*)
Sets z to z minus the product of x and y.

void **acb_inv** (acb_t *z*, const acb_t *x*, long *prec*)
Sets z to the multiplicative inverse of x.

void **acb_div_ui** (acb_t *z*, const acb_t *x*, ulong *y*, long *prec*)

void **acb_div_si** (acb_t *z*, const acb_t *x*, long *y*, long *prec*)

void **acb_div_fmpz** (acb_t *z*, const acb_t *x*, const fmpz_t *y*, long *prec*)

void **acb_div** (acb_t *z*, const acb_t *x*, const acb_t *y*, long *prec*)
Sets z to the quotient of x and y.

3.7.9 Elementary functions

void **acb_const_pi** (acb_t *y*, long *prec*)
Sets *y* to the constant π.

void **acb_log** (acb_t *y*, const acb_t *z*, long *prec*)
Sets *y* to the principal branch of the natural logarithm of *z*, computed as $\log(a+bi) = \frac{1}{2}\log(a^2+b^2)+i\arg(a+bi)$.

void **acb_log1p** (acb_t *z*, const acb_t *x*, long *prec*)
Sets $z = \log(1+x)$, computed accurately when $x \approx 0$.

void **acb_exp** (acb_t *y*, const acb_t *z*, long *prec*)
Sets *y* to the exponential function of *z*, computed as $\exp(a+bi) = \exp(a)(\cos(b)+\sin(b)i)$.

void **acb_exp_pi_i** (acb_t *y*, const acb_t *z*, long *prec*)
Sets *y* to $\exp(\pi i z)$.

void **acb_sin** (acb_t *s*, const acb_t *z*, long *prec*)

void **acb_cos** (acb_t *c*, const acb_t *z*, long *prec*)

void **acb_sin_cos** (arb_t *s*, arb_t *c*, const arb_t *z*, long *prec*)
Sets $s = \sin(z)$, $c = \cos(z)$, evaluated as $\sin(a+bi) = \sin(a)\cosh(b)+i\cos(a)\sinh(b)$, $\cos(a+bi) = \cos(a)\cosh(b)-i\sin(a)\sinh(b)$.

void **acb_tan** (acb_t *s*, const acb_t *z*, long *prec*)
Sets $s = \tan(z) = \sin(z)/\cos(z)$, evaluated as $\tan(a+bi) = \sin(2a)/(\cos(2a)+\cosh(2b)) + i\sinh(2b)/(\cos(2a)+\cosh(2b))$. If $|b|$ is small, the formula is evaluated as written; otherwise, we rewrite the hyperbolic functions in terms of decaying exponentials and evaluate the expression accurately using `arb_expm1()`.

void **acb_cot** (acb_t *s*, const acb_t *z*, long *prec*)
Sets $s = \cot(z) = \cos(z)/\sin(z)$, evaluated as $\cot(a+bi) = -\sin(2a)/(\cos(2a)-\cosh(2b)) + i\sinh(2b)/(\cos(2a)-\cosh(2b))$ using the same strategy as `acb_tan()`. If $|z|$ is close to zero, however, we evaluate $1/\tan(z)$ to avoid catastrophic cancellation.

void **acb_sin_pi** (acb_t *s*, const acb_t *z*, long *prec*)

void **acb_cos_pi** (acb_t *s*, const acb_t *z*, long *prec*)

void **acb_sin_cos_pi** (acb_t *s*, acb_t *c*, const acb_t *z*, long *prec*)
Sets $s = \sin(\pi z)$, $c = \cos(\pi z)$, evaluating the trigonometric factors of the real and imaginary part accurately via `arb_sin_cos_pi()`.

void **acb_tan_pi** (acb_t *s*, const acb_t *z*, long *prec*)
Sets $s = \tan(\pi z)$. Uses the same algorithm as `acb_tan()`, but evaluating the sine and cosine accurately via `arb_sin_cos_pi()`.

void **acb_cot_pi** (acb_t *s*, const acb_t *z*, long *prec*)
Sets $s = \cot(\pi z)$. Uses the same algorithm as `acb_cot()`, but evaluating the sine and cosine accurately via `arb_sin_cos_pi()`.

void **acb_atan** (acb_t *s*, const acb_t *z*, long *prec*)
Sets $s = \operatorname{atan}(z) = \frac{1}{2}i(\log(1-iz)-\log(1+iz))$.

void **acb_pow_fmpz** (acb_t *y*, const acb_t *b*, const fmpz_t *e*, long *prec*)

void **acb_pow_ui** (acb_t *y*, const acb_t *b*, ulong *e*, long *prec*)

void **acb_pow_si** (acb_t *y*, const acb_t *b*, long *e*, long *prec*)
Sets $y = b^e$ using binary exponentiation (with an initial division if $e < 0$). Note that these functions can get slow if the exponent is extremely large (in such cases `acb_pow()` may be superior).

void **acb_pow_arb** (acb_t *z*, const acb_t *x*, const arb_t *y*, long *prec*)

void **acb_pow** (acb_t *z*, const acb_t *x*, const acb_t *y*, long *prec*)
Sets $z = x^y$, computed using binary exponentiation if y if a small exact integer, as $z = (x^{1/2})^{2y}$ if y is a small exact half-integer, and generally as $z = \exp(y \log x)$.

void **acb_sqrt** (acb_t *r*, const acb_t *z*, long *prec*)
Sets *r* to the square root of *z*. If either the real or imaginary part is exactly zero, only a single real square root is needed. Generally, we use the formula $\sqrt{a+bi} = u/2 + ib/u, u = \sqrt{2(|a+bi|+a)}$, requiring two real square root extractions.

void **acb_rsqrt** (acb_t *r*, const acb_t *z*, long *prec*)
Sets *r* to the reciprocal square root of *z*. If either the real or imaginary part is exactly zero, only a single real reciprocal square root is needed. Generally, we use the formula $1/\sqrt{a+bi} = ((a+r)-bi)/v, r = |a+bi|, v = \sqrt{r|a+bi+r|^2}$, requiring one real square root and one real reciprocal square root.

3.7.10 Rising factorials

void **acb_rising_ui_bs** (acb_t *z*, const acb_t *x*, ulong *n*, long *prec*)

void **acb_rising_ui_rs** (acb_t *z*, const acb_t *x*, ulong *n*, ulong *step*, long *prec*)

void **acb_rising_ui_rec** (acb_t *z*, const acb_t *x*, ulong *n*, long *prec*)

void **acb_rising_ui** (acb_t *z*, const acb_t *x*, ulong *n*, long *prec*)
Computes the rising factorial $z = x(x+1)(x+2)\cdots(x+n-1)$.

The *bs* version uses binary splitting. The *rs* version uses rectangular splitting. The *rec* version uses either *bs* or *rs* depending on the input. The default version is currently identical to the *rec* version. In a future version, it will use the gamma function or asymptotic series when this is more efficient.

The *rs* version takes an optional *step* parameter for tuning purposes (to use the default step length, pass zero).

void **acb_rising2_ui_bs** (acb_t *u*, acb_t *v*, const acb_t *x*, ulong *n*, long *prec*)

void **acb_rising2_ui_rs** (acb_t *u*, acb_t *v*, const acb_t *x*, ulong *n*, ulong *step*, long *prec*)

void **acb_rising2_ui** (acb_t *u*, acb_t *v*, const acb_t *x*, ulong *n*, long *prec*)
Letting $u(x) = x(x+1)(x+2)\cdots(x+n-1)$, simultaneously compute $u(x)$ and $v(x) = u'(x)$, respectively using binary splitting, rectangular splitting (with optional nonzero step length *step* to override the default choice), and an automatic algorithm choice.

void **acb_rising_ui_get_mag** (mag_t *bound*, const acb_t *x*, ulong *n*)
Computes an upper bound for the absolute value of the rising factorial $z = x(x+1)(x+2)\cdots(x+n-1)$. Not currently optimized for large *n*.

3.7.11 Gamma function

void **acb_gamma** (acb_t *y*, const acb_t *x*, long *prec*)
Computes the gamma function $y = \Gamma(x)$.

void **acb_rgamma** (acb_t *y*, const acb_t *x*, long *prec*)
Computes the reciprocal gamma function $y = 1/\Gamma(x)$, avoiding division by zero at the poles of the gamma function.

void **acb_lgamma** (acb_t *y*, const acb_t *x*, long *prec*)
Computes the logarithmic gamma function $y = \log \Gamma(x)$.

The branch cut of the logarithmic gamma function is placed on the negative half-axis, which means that $\log \Gamma(z) + \log z = \log \Gamma(z+1)$ holds for all z, whereas $\log \Gamma(z) \neq \log(\Gamma(z))$ in general. Warning: this function does not currently use the reflection formula, and gets very slow for z far into the left half-plane.

void **acb_digamma** (acb_t *y*, const acb_t *x*, long *prec*)
Computes the digamma function $y = \psi(x) = (\log \Gamma(x))' = \Gamma'(x)/\Gamma(x)$.

3.7.12 Zeta function

void **acb_zeta** (acb_t *z*, const acb_t *s*, long *prec*)
Sets z to the value of the Riemann zeta function $\zeta(s)$. Note: for computing derivatives with respect to s, use `acb_poly_zeta_series()` or related methods.

void **acb_hurwitz_zeta** (acb_t *z*, const acb_t *s*, const acb_t *a*, long *prec*)
Sets z to the value of the Hurwitz zeta function $\zeta(s, a)$. Note: for computing derivatives with respect to s, use `acb_poly_zeta_series()` or related methods.

3.7.13 Polylogarithms

void **acb_polylog** (acb_t *w*, const acb_t *s*, const acb_t *z*, long *prec*)

void **acb_polylog_si** (acb_t *w*, long *s*, const acb_t *z*, long *prec*)
Sets *w* to the polylogarithm $\mathrm{Li}_s(z)$.

3.7.14 Arithmetic-geometric mean

void **acb_agm1** (acb_t *m*, const acb_t *z*, long *prec*)
Sets *m* to the arithmetic-geometric mean $M(z) = \mathrm{agm}(1, z)$, defined such that the function is continuous in the complex plane except for a branch cut along the negative half axis (where it is continuous from above). This corresponds to always choosing an "optimal" branch for the square root in the arithmetic-geometric mean iteration.

void **acb_agm1_cpx** (acb_ptr *m*, const acb_t *z*, long *len*, long *prec*)
Sets the coefficients in the array *m* to the power series expansion of the arithmetic-geometric mean at the point *z* truncated to length *len*, i.e. $M(z + x) \in \mathbb{C}[[x]]$.

3.8 acb_poly.h – polynomials over the complex numbers

An `acb_poly_t` represents a polynomial over the complex numbers, implemented as an array of coefficients of type `acb_struct`.

Most functions are provided in two versions: an underscore method which operates directly on pre-allocated arrays of coefficients and generally has some restrictions (such as requiring the lengths to be nonzero and not supporting aliasing of the input and output arrays), and a non-underscore method which performs automatic memory management and handles degenerate cases.

3.8.1 Types, macros and constants

acb_poly_struct

acb_poly_t

Contains a pointer to an array of coefficients (coeffs), the used length (length), and the allocated size of the array (alloc).

An *acb_poly_t* is defined as an array of length one of type *acb_poly_struct*, permitting an *acb_poly_t* to be passed by reference.

3.8.2 Memory management

void **acb_poly_init** (acb_poly_t *poly*)

Initializes the polynomial for use, setting it to the zero polynomial.

void **acb_poly_clear** (acb_poly_t *poly*)

Clears the polynomial, deallocating all coefficients and the coefficient array.

void **acb_poly_fit_length** (acb_poly_t *poly*, long *len*)

Makes sures that the coefficient array of the polynomial contains at least *len* initialized coefficients.

void **_acb_poly_set_length** (acb_poly_t *poly*, long *len*)

Directly changes the length of the polynomial, without allocating or deallocating coefficients. The value shold not exceed the allocation length.

void **_acb_poly_normalise** (acb_poly_t *poly*)

Strips any trailing coefficients which are identical to zero.

void **acb_poly_swap** (acb_poly_t *poly1*, acb_poly_t *poly2*)

Swaps *poly1* and *poly2* efficiently.

3.8.3 Basic properties and manipulation

long **acb_poly_length** (const acb_poly_t *poly*)

Returns the length of *poly*, i.e. zero if *poly* is identically zero, and otherwise one more than the index of the highest term that is not identically zero.

long **acb_poly_degree** (const acb_poly_t *poly*)

Returns the degree of *poly*, defined as one less than its length. Note that if one or several leading coefficients are balls containing zero, this value can be larger than the true degree of the exact polynomial represented by *poly*, so the return value of this function is effectively an upper bound.

void **acb_poly_zero** (acb_poly_t *poly*)

Sets *poly* to the zero polynomial.

void **acb_poly_one** (acb_poly_t *poly*)

Sets *poly* to the constant polynomial 1.

void **acb_poly_set** (acb_poly_t *dest*, const acb_poly_t *src*)

Sets *dest* to a copy of *src*.

void **acb_poly_set_round** (acb_poly_t *dest*, const acb_poly_t *src*, long *prec*)

Sets *dest* to a copy of *src*, rounded to *prec* bits.

void **acb_poly_set_coeff_si** (acb_poly_t *poly*, long *n*, long *c*)

void **acb_poly_set_coeff_acb** (acb_poly_t *poly*, long *n*, const acb_t *c*)

Sets the coefficient with index *n* in *poly* to the value *c*. We require that *n* is nonnegative.

void **acb_poly_get_coeff_acb** (acb_t *v*, const acb_poly_t *poly*, long *n*)

Sets *v* to the value of the coefficient with index *n* in *poly*. We require that *n* is nonnegative.

acb_poly_get_coeff_ptr (poly, n)
Given $n \geq 0$, returns a pointer to coefficient *n* of *poly*, or *NULL* if *n* exceeds the length of *poly*.

void **_acb_poly_shift_right** (acb_ptr *res*, acb_srcptr *poly*, long *len*, long *n*)

void **acb_poly_shift_right** (acb_poly_t *res*, const acb_poly_t *poly*, long *n*)
Sets *res* to *poly* divided by x^n, throwing away the lower coefficients. We require that *n* is nonnegative.

void **_acb_poly_shift_left** (acb_ptr *res*, acb_srcptr *poly*, long *len*, long *n*)

void **acb_poly_shift_left** (acb_poly_t *res*, const acb_poly_t *poly*, long *n*)
Sets *res* to *poly* multiplied by x^n. We require that *n* is nonnegative.

void **acb_poly_truncate** (acb_poly_t *poly*, long *n*)
Truncates *poly* to have length at most *n*, i.e. degree strictly smaller than *n*.

3.8.4 Input and output

void **acb_poly_printd** (const acb_poly_t *poly*, long *digits*)
Prints the polynomial as an array of coefficients, printing each coefficient using *arb_printd*.

3.8.5 Random generation

void **acb_poly_randtest** (acb_poly_t *poly*, flint_rand_t *state*, long *len*, long *prec*, long *mag_bits*)
Creates a random polynomial with length at most *len*.

3.8.6 Comparisons

int **acb_poly_equal** (const acb_poly_t *A*, const acb_poly_t *B*)
Returns nonzero iff *A* and *B* are identical as interval polynomials.

int **acb_poly_contains** (const acb_poly_t *poly1*, const acb_poly_t *poly2*)

int **acb_poly_contains_fmpz_poly** (const acb_poly_t *poly1*, const fmpz_poly_t *poly2*)

int **acb_poly_contains_fmpq_poly** (const acb_poly_t *poly1*, const fmpq_poly_t *poly2*)
Returns nonzero iff *poly2* is contained in *poly1*.

int **_acb_poly_overlaps** (acb_srcptr *poly1*, long *len1*, acb_srcptr *poly2*, long *len2*)

int **acb_poly_overlaps** (const acb_poly_t *poly1*, const acb_poly_t *poly2*)
Returns nonzero iff *poly1* overlaps with *poly2*. The underscore function requires that *len1* ist at least as large as *len2*.

int **acb_poly_get_unique_fmpz_poly** (fmpz_poly_t *z*, const acb_poly_t *x*)
If *x* contains a unique integer polynomial, sets *z* to that value and returns nonzero. Otherwise (if *x* represents no integers or more than one integer), returns zero, possibly partially modifying *z*.

int **acb_poly_is_real** (const acb_poly_t *poly*)
Returns nonzero iff all coefficients in *poly* have zero imaginary part.

3.8.7 Conversions

void **acb_poly_set_fmpz_poly** (acb_poly_t *poly*, const fmpz_poly_t *re*, long *prec*)

void **acb_poly_set2_fmpz_poly** (acb_poly_t *poly*, const fmpz_poly_t *re*, const fmpz_poly_t *im*, long *prec*) void **acb_poly_set_arb_poly**(acb_poly_t *poly*, const arb_poly_t *re*)

void **acb_poly_set2_arb_poly**(acb_poly_t *poly*, const arb_poly_t *re*, const arb_poly_t *im*)

void **acb_poly_set_fmpq_poly**(acb_poly_t *poly*, const fmpq_poly_t *re*, long *prec*)

void **acb_poly_set2_fmpq_poly**(acb_poly_t *poly*, const fmpq_poly_t *re*, const fmpq_poly_t *im*, long *prec*)
Sets *poly* to the given real part *re* plus the imaginary part *im*, both rounded to *prec* bits.

void **acb_poly_set_acb**(acb_poly_t *poly*, long *src*)

void **acb_poly_set_si**(acb_poly_t *poly*, long *src*)
Sets *poly* to *src*.

3.8.8 Arithmetic

void **_acb_poly_add**(acb_ptr *C*, acb_srcptr *A*, long *lenA*, acb_srcptr *B*, long *lenB*, long *prec*)
Sets *{C, max(lenA, lenB)}* to the sum of *{A, lenA}* and *{B, lenB}*. Allows aliasing of the input and output operands.

void **acb_poly_add**(acb_poly_t *C*, const acb_poly_t *A*, const acb_poly_t *B*, long *prec*)
Sets *C* to the sum of *A* and *B*.

void **_acb_poly_sub**(acb_ptr *C*, acb_srcptr *A*, long *lenA*, acb_srcptr *B*, long *lenB*, long *prec*)
Sets *{C, max(lenA, lenB)}* to the difference of *{A, lenA}* and *{B, lenB}*. Allows aliasing of the input and output operands.

void **acb_poly_sub**(acb_poly_t *C*, const acb_poly_t *A*, const acb_poly_t *B*, long *prec*)
Sets *C* to the difference of *A* and *B*.

void **acb_poly_neg**(acb_poly_t *C*, const acb_poly_t *A*)
Sets *C* to the negation of *A*.

void **acb_poly_scalar_mul_2exp_si**(acb_poly_t *C*, const acb_poly_t *A*, long *c*)
Sets *C* to *A* multiplied by 2^c.

void **_acb_poly_mullow_classical**(acb_ptr *C*, acb_srcptr *A*, long *lenA*, acb_srcptr *B*, long *lenB*, long *n*, long *prec*)

void **_acb_poly_mullow_transpose**(acb_ptr *C*, acb_srcptr *A*, long *lenA*, acb_srcptr *B*, long *lenB*, long *n*, long *prec*)

void **_acb_poly_mullow_transpose_gauss**(acb_ptr *C*, acb_srcptr *A*, long *lenA*, acb_srcptr *B*, long *lenB*, long *n*, long *prec*)

void **_acb_poly_mullow**(acb_ptr *C*, acb_srcptr *A*, long *lenA*, acb_srcptr *B*, long *lenB*, long *n*, long *prec*)
Sets *{C, n}* to the product of *{A, lenA}* and *{B, lenB}*, truncated to length *n*. The output is not allowed to be aliased with either of the inputs. We require $\mathrm{lenA} \ge \mathrm{lenB} > 0$, $n > 0$, $\mathrm{lenA} + \mathrm{lenB} - 1 \ge n$.

The *classical* version uses a plain loop.

The *transpose* version evaluates the product using four real polynomial multiplications (via `_arb_poly_mullow()`).

The *transpose_gauss* version evaluates the product using three real polynomial multiplications. This is almost always faster than *transpose*, but has worse numerical stability when the coefficients vary in magnitude.

The default function `_acb_poly_mullow()` automatically switches been *classical* and *transpose* multiplication.

If the input pointers are identical (and the lengths are the same), they are assumed to represent the same polynomial, and its square is computed.

void **acb_poly_mullow_classical** (acb_poly_t *C*, const acb_poly_t *A*, const acb_poly_t *B*, long *n*, long *prec*)

void **acb_poly_mullow_transpose** (acb_poly_t *C*, const acb_poly_t *A*, const acb_poly_t *B*, long *n*, long *prec*)

void **acb_poly_mullow_transpose_gauss** (acb_poly_t *C*, const acb_poly_t *A*, const acb_poly_t *B*, long *n*, long *prec*)

void **acb_poly_mullow** (acb_poly_t *C*, const acb_poly_t *A*, const acb_poly_t *B*, long *n*, long *prec*)
Sets *C* to the product of *A* and *B*, truncated to length *n*. If the same variable is passed for *A* and *B*, sets *C* to the square of *A* truncated to length *n*.

void **_acb_poly_mul** (acb_ptr *C*, acb_srcptr *A*, long *lenA*, acb_srcptr *B*, long *lenB*, long *prec*)
Sets *{C, lenA + lenB - 1}* to the product of *{A, lenA}* and *{B, lenB}*. The output is not allowed to be aliased with either of the inputs. We require $\mathrm{lenA} \geq \mathrm{lenB} > 0$. This function is implemented as a simple wrapper for `_acb_poly_mullow()`.

If the input pointers are identical (and the lengths are the same), they are assumed to represent the same polynomial, and its square is computed.

void **acb_poly_mul** (acb_poly_t *C*, const acb_poly_t *A1*, const acb_poly_t *B2*, long *prec*)
Sets *C* to the product of *A* and *B*. If the same variable is passed for *A* and *B*, sets *C* to the square of *A*.

void **_acb_poly_inv_series** (acb_ptr *Qinv*, acb_srcptr *Q*, long *Qlen*, long *len*, long *prec*)
Sets *{Qinv, len}* to the power series inverse of *{Q, Qlen}*. Uses Newton iteration.

void **acb_poly_inv_series** (acb_poly_t *Qinv*, const acb_poly_t *Q*, long *n*, long *prec*)
Sets *Qinv* to the power series inverse of *Q*.

void **_acb_poly_div_series** (acb_ptr *Q*, acb_srcptr *A*, long *Alen*, acb_srcptr *B*, long *Blen*, long *n*, long *prec*)
Sets *{Q, n}* to the power series quotient of *{A, Alen}* by *{B, Blen}*. Uses Newton iteration followed by multiplication.

void **acb_poly_div_series** (acb_poly_t *Q*, const acb_poly_t *A*, const acb_poly_t *B*, long *n*, long *prec*)
Sets *Q* to the power series quotient *A* divided by *B*, truncated to length *n*.

void **_acb_poly_div** (acb_ptr *Q*, acb_srcptr *A*, long *lenA*, acb_srcptr *B*, long *lenB*, long *prec*)

void **_acb_poly_rem** (acb_ptr *R*, acb_srcptr *A*, long *lenA*, acb_srcptr *B*, long *lenB*, long *prec*)

void **_acb_poly_divrem** (acb_ptr *Q*, acb_ptr *R*, acb_srcptr *A*, long *lenA*, acb_srcptr *B*, long *lenB*, long *prec*)

void **acb_poly_divrem** (acb_poly_t *Q*, acb_poly_t *R*, const acb_poly_t *A*, const acb_poly_t *B*, long *prec*)
Performs polynomial division with remainder, computing a quotient Q and a remainder R such that $A = BQ + R$. The implementation reverses the inputs and performs power series division.

If the leading coefficient of B contains zero (or if B is identically zero), returns 0 indicating failure without modifying the outputs. Otherwise returns nonzero.

void **_acb_poly_div_root** (acb_ptr *Q*, acb_t *R*, acb_srcptr *A*, long *len*, const acb_t *c*, long *prec*)
Divides A by the polynomial $x - c$, computing the quotient Q as well as the remainder $R = f(c)$.

3.8.9 Composition

void **_acb_poly_compose_horner** (acb_ptr *res*, acb_srcptr *poly1*, long *len1*, acb_srcptr *poly2*, long *len2*, long *prec*)

void **acb_poly_compose_horner** (acb_poly_t *res*, const acb_poly_t *poly1*, const acb_poly_t *poly2*, long *prec*) void **_acb_poly_compose_divconquer** (acb_ptr *res*, acb_srcptr *poly1*, long *len1*, acb_srcptr *poly2*, long *len2*, long *prec*)

void **acb_poly_compose_divconquer** (acb_poly_t *res*, const acb_poly_t *poly1*, const acb_poly_t *poly2*, long *prec*)

void **_acb_poly_compose** (acb_ptr *res*, acb_srcptr *poly1*, long *len1*, acb_srcptr *poly2*, long *len2*, long *prec*)

void **acb_poly_compose** (acb_poly_t *res*, const acb_poly_t *poly1*, const acb_poly_t *poly2*, long *prec*)

Sets *res* to the composition $h(x) = f(g(x))$ where f is given by *poly1* and g is given by *poly2*, respectively using Horner's rule, divide-and-conquer, and an automatic choice between the two algorithms. The underscore methods do not support aliasing of the output with either input polynomial.

void **_acb_poly_compose_series_horner** (acb_ptr *res*, acb_srcptr *poly1*, long *len1*, acb_srcptr *poly2*, long *len2*, long *n*, long *prec*)

void **acb_poly_compose_series_horner** (acb_poly_t *res*, const acb_poly_t *poly1*, const acb_poly_t *poly2*, long *n*, long *prec*)

void **_acb_poly_compose_series_brent_kung** (acb_ptr *res*, acb_srcptr *poly1*, long *len1*, acb_srcptr *poly2*, long *len2*, long *n*, long *prec*)

void **acb_poly_compose_series_brent_kung** (acb_poly_t *res*, const acb_poly_t *poly1*, const acb_poly_t *poly2*, long *n*, long *prec*)

void **_acb_poly_compose_series** (acb_ptr *res*, acb_srcptr *poly1*, long *len1*, acb_srcptr *poly2*, long *len2*, long *n*, long *prec*)

void **acb_poly_compose_series** (acb_poly_t *res*, const acb_poly_t *poly1*, const acb_poly_t *poly2*, long *n*, long *prec*)

Sets *res* to the power series composition $h(x) = f(g(x))$ truncated to order $O(x^n)$ where f is given by *poly1* and g is given by *poly2*, respectively using Horner's rule, the Brent-Kung baby step-giant step algorithm, and an automatic choice between the two algorithms. We require that the constant term in $g(x)$ is exactly zero. The underscore methods do not support aliasing of the output with either input polynomial.

void **_acb_poly_revert_series_lagrange** (acb_ptr *h*, acb_srcptr *f*, long *flen*, long *n*, long *prec*)

void **acb_poly_revert_series_lagrange** (acb_poly_t *h*, const acb_poly_t *f*, long *n*, long *prec*)

void **_acb_poly_revert_series_newton** (acb_ptr *h*, acb_srcptr *f*, long *flen*, long *n*, long *prec*)

void **acb_poly_revert_series_newton** (acb_poly_t *h*, const acb_poly_t *f*, long *n*, long *prec*)

void **_acb_poly_revert_series_lagrange_fast** (acb_ptr *h*, acb_srcptr *f*, long *flen*, long *n*, long *prec*)

void **acb_poly_revert_series_lagrange_fast** (acb_poly_t *h*, const acb_poly_t *f*, long *n*, long *prec*)

void **_acb_poly_revert_series** (acb_ptr *h*, acb_srcptr *f*, long *flen*, long *n*, long *prec*)

void **acb_poly_revert_series** (acb_poly_t *h*, const acb_poly_t *f*, long *n*, long *prec*)

Sets h to the power series reversion of f, i.e. the expansion of the compositional inverse function $f^{-1}(x)$, truncated to order $O(x^n)$, using respectively Lagrange inversion, Newton iteration, fast Lagrange inversion, and a default algorithm choice.

We require that the constant term in f is exactly zero and that the linear term is nonzero. The underscore methods assume that *flen* is at least 2, and do not support aliasing.

3.8.10 Evaluation

void **_acb_poly_evaluate_horner** (acb_t *y*, acb_srcptr *f*, long *len*, const acb_t *x*, long *prec*)

void **acb_poly_evaluate_horner** (acb_t *y*, const acb_poly_t *f*, const acb_t *x*, long *prec*) void **_acb_poly_evaluate_rectangular** (acb_t *y*, acb_srcptr *f*, long *len*, const acb_t *x*, long *prec*)

void **acb_poly_evaluate_rectangular** (acb_t *y*, const acb_poly_t *f*, const acb_t *x*, long *prec*)

void **_acb_poly_evaluate** (acb_t *y*, acb_srcptr *f*, long *len*, const acb_t *x*, long *prec*)

void **acb_poly_evaluate** (acb_t *y*, const acb_poly_t *f*, const acb_t *x*, long *prec*)

Sets $y = f(x)$, evaluated respectively using Horner's rule, rectangular splitting, and an automatic algorithm choice.

void **_acb_poly_evaluate2_horner** (acb_t *y*, acb_t *z*, acb_srcptr *f*, long *len*, const acb_t *x*, long *prec*)

void **acb_poly_evaluate2_horner** (acb_t *y*, acb_t *z*, const acb_poly_t *f*, const acb_t *x*, long *prec*)

void **_acb_poly_evaluate2_rectangular** (acb_t *y*, acb_t *z*, acb_srcptr *f*, long *len*, const acb_t *x*, long *prec*)

void **acb_poly_evaluate2_rectangular** (acb_t *y*, acb_t *z*, const acb_poly_t *f*, const acb_t *x*, long *prec*)

void **_acb_poly_evaluate2** (acb_t *y*, acb_t *z*, acb_srcptr *f*, long *len*, const acb_t *x*, long *prec*)

void **acb_poly_evaluate2** (acb_t *y*, acb_t *z*, const acb_poly_t *f*, const acb_t *x*, long *prec*)

Sets $y = f(x), z = f'(x)$, evaluated respectively using Horner's rule, rectangular splitting, and an automatic algorithm choice.

When Horner's rule is used, the only advantage of evaluating the function and its derivative simultaneously is that one does not have to generate the derivative polynomial explicitly. With the rectangular splitting algorithm, the powers can be reused, making simultaneous evaluation slightly faster.

3.8.11 Product trees

void **_acb_poly_product_roots** (acb_ptr *poly*, acb_srcptr *xs*, long *n*, long *prec*)

void **acb_poly_product_roots** (acb_poly_t *poly*, acb_srcptr *xs*, long *n*, long *prec*)

Generates the polynomial $(x - x_0)(x - x_1) \cdots (x - x_{n-1})$.

acb_ptr * **_acb_poly_tree_alloc** (long *len*)

Returns an initialized data structured capable of representing a remainder tree (product tree) of *len* roots.

void **_acb_poly_tree_free** (acb_ptr * *tree*, long *len*)

Deallocates a tree structure as allocated using *_acb_poly_tree_alloc*.

void **_acb_poly_tree_build** (acb_ptr * *tree*, acb_srcptr *roots*, long *len*, long *prec*)

Constructs a product tree from a given array of *len* roots. The tree structure must be pre-allocated to the specified length using `_acb_poly_tree_alloc()`.

3.8.12 Multipoint evaluation

void **_acb_poly_evaluate_vec_iter** (acb_ptr *ys*, acb_srcptr *poly*, long *plen*, acb_srcptr *xs*, long *n*, long *prec*)

void **acb_poly_evaluate_vec_iter** (acb_ptr *ys*, const acb_poly_t *poly*, acb_srcptr *xs*, long *n*, long *prec*)

Evaluates the polynomial simultaneously at *n* given points, calling `_acb_poly_evaluate()` repeatedly.

void **_acb_poly_evaluate_vec_fast_precomp** (acb_ptr *vs*, acb_srcptr *poly*, long *plen*, acb_ptr * *tree*, long *len*, long *prec*)

void **_acb_poly_evaluate_vec_fast** (acb_ptr *ys*, acb_srcptr *poly*, long *plen*, acb_srcptr *xs*, long *n*, long *prec*)

void **acb_poly_evaluate_vec_fast** (acb_ptr *ys*, const acb_poly_t *poly*, acb_srcptr *xs*, long *n*, long *prec*)
Evaluates the polynomial simultaneously at *n* given points, using fast multipoint evaluation.

3.8.13 Interpolation

void **_acb_poly_interpolate_newton** (acb_ptr *poly*, acb_srcptr *xs*, acb_srcptr *ys*, long *n*, long *prec*)

void **acb_poly_interpolate_newton** (acb_poly_t *poly*, acb_srcptr *xs*, acb_srcptr *ys*, long *n*, long *prec*)
Recovers the unique polynomial of length at most *n* that interpolates the given *x* and *y* values. This implementation first interpolates in the Newton basis and then converts back to the monomial basis.

void **_acb_poly_interpolate_barycentric** (acb_ptr *poly*, acb_srcptr *xs*, acb_srcptr *ys*, long *n*, long *prec*)

void **acb_poly_interpolate_barycentric** (acb_poly_t *poly*, acb_srcptr *xs*, acb_srcptr *ys*, long *n*, long *prec*)
Recovers the unique polynomial of length at most *n* that interpolates the given *x* and *y* values. This implementation uses the barycentric form of Lagrange interpolation.

void **_acb_poly_interpolation_weights** (acb_ptr *w*, acb_ptr * *tree*, long *len*, long *prec*)

void **_acb_poly_interpolate_fast_precomp** (acb_ptr *poly*, acb_srcptr *ys*, acb_ptr * *tree*, acb_srcptr *weights*, long *len*, long *prec*)

void **_acb_poly_interpolate_fast** (acb_ptr *poly*, acb_srcptr *xs*, acb_srcptr *ys*, long *len*, long *prec*)

void **acb_poly_interpolate_fast** (acb_poly_t *poly*, acb_srcptr *xs*, acb_srcptr *ys*, long *n*, long *prec*)
Recovers the unique polynomial of length at most *n* that interpolates the given *x* and *y* values, using fast Lagrange interpolation. The precomp function takes a precomputed product tree over the *x* values and a vector of interpolation weights as additional inputs.

3.8.14 Differentiation

void **_acb_poly_derivative** (acb_ptr *res*, acb_srcptr *poly*, long *len*, long *prec*)
Sets *{res, len - 1}* to the derivative of *{poly, len}*. Allows aliasing of the input and output.

void **acb_poly_derivative** (acb_poly_t *res*, const acb_poly_t *poly*, long *prec*)
Sets *res* to the derivative of *poly*.

void **_acb_poly_integral** (acb_ptr *res*, acb_srcptr *poly*, long *len*, long *prec*)
Sets *{res, len}* to the integral of *{poly, len - 1}*. Allows aliasing of the input and output.

void **acb_poly_integral** (acb_poly_t *res*, const acb_poly_t *poly*, long *prec*)
Sets *res* to the integral of *poly*.

3.8.15 Elementary functions

void **_acb_poly_pow_ui_trunc_binexp** (acb_ptr *res*, acb_srcptr *f*, long *flen*, ulong *exp*, long *len*, long *prec*)
Sets *{res, len}* to *{f, flen}* raised to the power *exp*, truncated to length *len*. Requires that *len* is no longer than the length of the power as computed without truncation (i.e. no zero-padding is performed). Does not support aliasing of the input and output, and requires that *flen* and *len* are positive. Uses binary expontiation.

void **acb_poly_pow_ui_trunc_binexp** (acb_poly_t *res*, const acb_poly_t *poly*, ulong *exp*, long *len*, long *prec*)
Sets *res* to *poly* raised to the power *exp*, truncated to length *len*. Uses binary exponentiation.

void **_acb_poly_pow_ui** (acb_ptr *res*, acb_srcptr *f*, long *flen*, ulong *exp*, long *prec*)
Sets *res* to *{f, flen}* raised to the power *exp*. Does not support aliasing of the input and output, and requires that *flen* is positive.

void **acb_poly_pow_ui** (acb_poly_t *res*, const acb_poly_t *poly*, ulong *exp*, long *prec*)
Sets *res* to *poly* raised to the power *exp*.

void **_acb_poly_pow_series** (acb_ptr *h*, acb_srcptr *f*, long *flen*, acb_srcptr *g*, long *glen*, long *len*, long *prec*)
Sets *{h, len}* to the power series $f(x)^{g(x)} = \exp(g(x) \log f(x))$ truncated to length *len*. This function detects special cases such as *g* being an exact small integer or $\pm 1/2$, and computes such powers more efficiently. This function does not support aliasing of the output with either of the input operands. It requires that all lengths are positive, and assumes that *flen* and *glen* do not exceed *len*.

void **acb_poly_pow_series** (acb_poly_t *h*, const acb_poly_t *f*, const acb_poly_t *g*, long *len*, long *prec*)
Sets *h* to the power series $f(x)^{g(x)} = \exp(g(x) \log f(x))$ truncated to length *len*. This function detects special cases such as *g* being an exact small integer or $\pm 1/2$, and computes such powers more efficiently.

void **_acb_poly_pow_acb_series** (acb_ptr *h*, acb_srcptr *f*, long *flen*, const acb_t *g*, long *len*, long *prec*)
Sets *{h, len}* to the power series $f(x)^g = \exp(g \log f(x))$ truncated to length *len*. This function detects special cases such as *g* being an exact small integer or $\pm 1/2$, and computes such powers more efficiently. This function does not support aliasing of the output with either of the input operands. It requires that all lengths are positive, and assumes that *flen* does not exceed *len*.

void **acb_poly_pow_acb_series** (acb_poly_t *h*, const acb_poly_t *f*, const acb_t *g*, long *len*, long *prec*)
Sets *h* to the power series $f(x)^g = \exp(g \log f(x))$ truncated to length *len*.

void **_acb_poly_sqrt_series** (acb_ptr *g*, acb_srcptr *h*, long *hlen*, long *n*, long *prec*)

void **acb_poly_sqrt_series** (acb_poly_t *g*, const acb_poly_t *h*, long *n*, long *prec*)
Sets *g* to the power series square root of *h*, truncated to length *n*. Uses division-free Newton iteration for the reciprocal square root, followed by a multiplication.

The underscore method does not support aliasing of the input and output arrays. It requires that *hlen* and *n* are greater than zero.

void **_acb_poly_rsqrt_series** (acb_ptr *g*, acb_srcptr *h*, long *hlen*, long *n*, long *prec*)

void **acb_poly_rsqrt_series** (acb_poly_t *g*, const acb_poly_t *h*, long *n*, long *prec*)
Sets *g* to the reciprocal power series square root of *h*, truncated to length *n*. Uses division-free Newton iteration.

The underscore method does not support aliasing of the input and output arrays. It requires that *hlen* and *n* are greater than zero.

void **_acb_poly_log_series** (acb_ptr *res*, acb_srcptr *f*, long *flen*, long *n*, long *prec*)

void **acb_poly_log_series** (acb_poly_t *res*, const acb_poly_t *f*, long *n*, long *prec*)
Sets *res* to the power series logarithm of *f*, truncated to length *n*. Uses the formula $\log(f(x)) = \int f'(x)/f(x)dx$, adding the logarithm of the constant term in *f* as the constant of integration.

The underscore method supports aliasing of the input and output arrays. It requires that *flen* and *n* are greater than zero.

void **_acb_poly_atan_series** (acb_ptr *res*, acb_srcptr *f*, long *flen*, long *n*, long *prec*)

void **acb_poly_atan_series** (acb_poly_t *res*, const acb_poly_t *f*, long *n*, long *prec*)
Sets *res* the power series inverse tangent of *f*, truncated to length *n*.

Uses the formula

$$\tan^{-1}(f(x)) = \int f'(x)/(1 + f(x)^2)dx,$$

adding the function of the constant term in *f* as the constant of integration.

The underscore method supports aliasing of the input and output arrays. It requires that *flen* and *n* are greater than zero.

void **_acb_poly_exp_series_basecase** (acb_ptr *f*, acb_srcptr *h*, long *hlen*, long *n*, long *prec*)

void **acb_poly_exp_series_basecase** (acb_poly_t *f*, const acb_poly_t *h*, long *n*, long *prec*)

void **_acb_poly_exp_series** (acb_ptr *f*, acb_srcptr *h*, long *hlen*, long *n*, long *prec*)

void **acb_poly_exp_series** (acb_poly_t *f*, const acb_poly_t *h*, long *n*, long *prec*)

Sets f to the power series exponential of h, truncated to length n.

The basecase version uses a simple recurrence for the coefficients, requiring $O(nm)$ operations where m is the length of h.

The main implementation uses Newton iteration, starting from a small number of terms given by the basecase algorithm. The complexity is $O(M(n))$. Redundant operations in the Newton iteration are avoided by using the scheme described in [HZ2004].

The underscore methods support aliasing and allow the input to be shorter than the output, but require the lengths to be nonzero.

void **_acb_poly_sin_cos_series_basecase** (acb_ptr *s*, acb_ptr *c*, acb_srcptr *h*, long *hlen*, long *n*, long *prec*)

void **acb_poly_sin_cos_series_basecase** (acb_poly_t *s*, acb_poly_t *c*, const acb_poly_t *h*, long *n*, long *prec*)

void **_acb_poly_sin_cos_series_tangent** (acb_ptr *s*, acb_ptr *c*, acb_srcptr *h*, long *hlen*, long *n*, long *prec*)

void **acb_poly_sin_cos_series_tangent** (acb_poly_t *s*, acb_poly_t *c*, const acb_poly_t *h*, long *n*, long *prec*)

void **_acb_poly_sin_cos_series** (acb_ptr *s*, acb_ptr *c*, acb_srcptr *h*, long *hlen*, long *n*, long *prec*)

void **acb_poly_sin_cos_series** (acb_poly_t *s*, acb_poly_t *c*, const acb_poly_t *h*, long *n*, long *prec*)

Sets *s* and *c* to the power series sine and cosine of *h*, computed simultaneously.

The *basecase* version uses a simple recurrence for the coefficients, requiring $O(nm)$ operations where m is the length of h.

The *tangent* version uses the tangent half-angle formulas to compute the sine and cosine via `_acb_poly_tan_series()`. This requires $O(M(n))$ operations. When $h = h_0 + h_1$ where the constant term h_0 is nonzero, the evaluation is done as $\sin(h_0 + h_1) = \cos(h_0)\sin(h_1) + \sin(h_0)\cos(h_1)$, $\cos(h_0 + h_1) = \cos(h_0)\cos(h_1) - \sin(h_0)\sin(h_1)$, to improve accuracy and avoid dividing by zero at the poles of the tangent function.

The default version automatically selects between the *basecase* and *tangent* algorithms depending on the input.

The underscore methods support aliasing and require the lengths to be nonzero.

void **_acb_poly_sin_series** (acb_ptr *s*, acb_srcptr *h*, long *hlen*, long *n*, long *prec*)

void **acb_poly_sin_series** (acb_poly_t *s*, const acb_poly_t *h*, long *n*, long *prec*)

void **_acb_poly_cos_series** (acb_ptr *c*, acb_srcptr *h*, long *hlen*, long *n*, long *prec*)

void **acb_poly_cos_series** (acb_poly_t *c*, const acb_poly_t *h*, long *n*, long *prec*)

Respectively evaluates the power series sine or cosine. These functions simply wrap `_acb_poly_sin_cos_series()`. The underscore methods support aliasing and require the lengths to be nonzero.

void **_acb_poly_tan_series** (acb_ptr *g*, acb_srcptr *h*, long *hlen*, long *len*, long *prec*)

void **acb_poly_tan_series** (acb_poly_t *g*, const acb_poly_t *h*, long *n*, long *prec*)
Sets g to the power series tangent of h.

For small *n* takes the quotient of the sine and cosine as computed using the basecase algorithm. For large *n*, uses Newton iteration to invert the inverse tangent series. The complexity is $O(M(n))$.

The underscore version does not support aliasing, and requires the lengths to be nonzero.

3.8.16 Gamma function

void **_acb_poly_gamma_series** (acb_ptr *res*, acb_srcptr *h*, long *hlen*, long *n*, long *prec*)

void **acb_poly_gamma_series** (acb_poly_t *res*, const acb_poly_t *h*, long *n*, long *prec*)

void **_acb_poly_rgamma_series** (acb_ptr *res*, acb_srcptr *h*, long *hlen*, long *n*, long *prec*)

void **acb_poly_rgamma_series** (acb_poly_t *res*, const acb_poly_t *h*, long *n*, long *prec*)

void **_acb_poly_lgamma_series** (acb_ptr *res*, acb_srcptr *h*, long *hlen*, long *n*, long *prec*)

void **acb_poly_lgamma_series** (acb_poly_t *res*, const acb_poly_t *h*, long *n*, long *prec*)
Sets *res* to the series expansion of $\Gamma(h(x))$, $1/\Gamma(h(x))$, or $\log \Gamma(h(x))$, truncated to length *n*.

These functions first generate the Taylor series at the constant term of *h*, and then call `_acb_poly_compose_series()`. The Taylor coefficients are generated using Stirling's series.

The underscore methods support aliasing of the input and output arrays, and require that *hlen* and *n* are greater than zero.

void **_acb_poly_rising_ui_series** (acb_ptr *res*, acb_srcptr *f*, long *flen*, ulong *r*, long *trunc*, long *prec*)

void **acb_poly_rising_ui_series** (acb_poly_t *res*, const acb_poly_t *f*, ulong *r*, long *trunc*, long *prec*)
Sets *res* to the rising factorial $(f)(f+1)(f+2)\cdots(f+r-1)$, truncated to length *trunc*. The underscore method assumes that *flen*, *r* and *trunc* are at least 1, and does not support aliasing. Uses binary splitting.

3.8.17 Power sums

void **_acb_poly_powsum_series_naive** (acb_ptr *z*, const acb_t *s*, const acb_t *a*, const acb_t *q*, long *n*, long *len*, long *prec*)

void **_acb_poly_powsum_series_naive_threaded** (acb_ptr *z*, const acb_t *s*, const acb_t *a*, const acb_t *q*, long *n*, long *len*, long *prec*)
Computes

$$z = S(s,a,n) = \sum_{k=0}^{n-1} \frac{q^k}{(k+a)^{s+t}}$$

as a power series in t truncated to length *len*. This function evaluates the sum naively term by term. The *threaded* version splits the computation over the number of threads returned by *flint_get_num_threads()*.

void **_acb_poly_powsum_one_series_sieved** (acb_ptr *z*, const acb_t *s*, long *n*, long *len*, long *prec*)
Computes

$$z = S(s,1,n) \sum_{k=1}^{n} \frac{1}{k^{s+t}}$$

as a power series in t truncated to length *len*. This function stores a table of powers that have already been calculated, computing $(ij)^r$ as $i^r j^r$ whenever $k = ij$ is composite. As a further optimization, it groups all even k and evaluates the sum as a polynomial in $2^{-(s+t)}$. This scheme requires about $n/\log n$ powers, $n/2$

multiplications, and temporary storage of $n/6$ power series. Due to the extra power series multiplications, it is only faster than the naive algorithm when *len* is small.

3.8.18 Zeta function

void **_acb_poly_zeta_em_choose_param**(arf_t *bound*, ulong * *N*, ulong * *M*, const acb_t *s*, const acb_t *a*, long *d*, long *target*, long *prec*)
Chooses *N* and *M* for Euler-Maclaurin summation of the Hurwitz zeta function, using a default algorithm.

void **_acb_poly_zeta_em_bound1**(arf_t *bound*, const acb_t *s*, const acb_t *a*, long *N*, long *M*, long *d*, long *wp*)

void **_acb_poly_zeta_em_bound**(arb_ptr *vec*, const acb_t *s*, const acb_t *a*, ulong *N*, ulong *M*, long *d*, long *wp*)
Compute bounds for Euler-Maclaurin evaluation of the Hurwitz zeta function or its power series, using the formulas in [Joh2013].

void **_acb_poly_zeta_em_tail_naive**(acb_ptr *z*, const acb_t *s*, const acb_t *Na*, acb_srcptr *Nasx*, long *M*, long *len*, long *prec*)

void **_acb_poly_zeta_em_tail_bsplit**(acb_ptr *z*, const acb_t *s*, const acb_t *Na*, acb_srcptr *Nasx*, long *M*, long *len*, long *prec*)
Evaluates the tail in the Euler-Maclaurin sum for the Hurwitz zeta function, respectively using the naive recurrence and binary splitting.

void **_acb_poly_zeta_em_sum**(acb_ptr *z*, const acb_t *s*, const acb_t *a*, int *deflate*, ulong *N*, ulong *M*, long *d*, long *prec*)
Evaluates the truncated Euler-Maclaurin sum of order N, M for the length-*d* truncated Taylor series of the Hurwitz zeta function $\zeta(s, a)$ at s, using a working precision of *prec* bits. With $a = 1$, this gives the usual Riemann zeta function.

If *deflate* is nonzero, $\zeta(s, a) - 1/(s - 1)$ is evaluated (which permits series expansion at $s = 1$).

void **_acb_poly_zeta_cpx_series**(acb_ptr *z*, const acb_t *s*, const acb_t *a*, int *deflate*, long *d*, long *prec*)
Computes the series expansion of $\zeta(s + x, a)$ (or $\zeta(s + x, a) - 1/(s + x - 1)$ if *deflate* is nonzero) to order *d*.

This function wraps `_acb_poly_zeta_em_sum()`, automatically choosing default values for N, M using `_acb_poly_zeta_em_choose_param()` to target an absolute truncation error of $2^{-\operatorname{prec}}$.

void **_acb_poly_zeta_series**(acb_ptr *res*, acb_srcptr *h*, long *hlen*, const acb_t *a*, int *deflate*, long *len*, long *prec*)

void **acb_poly_zeta_series**(acb_poly_t *res*, const acb_poly_t *f*, const acb_t *a*, int *deflate*, long *n*, long *prec*)
Sets *res* to the Hurwitz zeta function $\zeta(s, a)$ where s a power series and a is a constant, truncated to length *n*. To evaluate the usual Riemann zeta function, set $a = 1$.

If *deflate* is nonzero, evaluates $\zeta(s, a) + 1/(1 - s)$, which is well-defined as a limit when the constant term of s is 1. In particular, expanding $\zeta(s, a) + 1/(1 - s)$ with $s = 1 + x$ gives the Stieltjes constants

$$\sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \gamma_k(a) x^k.$$

If $a = 1$, this implementation uses the reflection formula if the midpoint of the constant term of s is negative.

3.8.19 Other special functions

void **_acb_poly_polylog_cpx_small**(acb_ptr *w*, const acb_t *s*, const acb_t *z*, long *len*, long *prec*)

void **_acb_poly_polylog_cpx_zeta**(acb_ptr *w*, const acb_t *s*, const acb_t *z*, long *len*, long *prec*)

void **_acb_poly_polylog_cpx** (acb_ptr *w*, const acb_t *s*, const acb_t *z*, long *len*, long *prec*)
Sets *w* to the Taylor series with respect to *x* of the polylogarithm $\mathrm{Li}_{s+x}(z)$, where *s* and *z* are given complex constants. The output is computed to length *len* which must be positive. Aliasing between *w* and *s* or *z* is not permitted.

The *small* version uses the standard power series expansion with respect to *z*, convergent when $|z| < 1$. The *zeta* version evaluates the polylogarithm as a sum of two Hurwitz zeta functions. The default version automatically delegates to the *small* version when *z* is close to zero, and the *zeta* version otherwise. For further details, see *Algorithms for polylogarithms*.

void **_acb_poly_polylog_series** (acb_ptr *w*, acb_srcptr *s*, long *slen*, const acb_t *z*, long *len*, long *prec*)

void **acb_poly_polylog_series** (acb_poly_t *w*, const acb_poly_t *s*, const acb_t *z*, long *len*, long *prec*)
Sets *w* to the polylogarithm $\mathrm{Li}_s(z)$ where *s* is a given power series, truncating the output to length *len*. The underscore method requires all lengths to be positive and supports aliasing between all inputs and outputs.

void **_acb_poly_erf_series** (acb_ptr *res*, acb_srcptr *z*, long *zlen*, long *n*, long *prec*)

void **acb_poly_erf_series** (acb_poly_t *res*, const acb_poly_t *z*, long *n*, long *prec*)
Sets *res* to the error function of the power series *z*, truncated to length *n*.

void **_acb_poly_upper_gamma_series** (acb_ptr *res*, acb_t *s*, acb_srcptr *z*, long *zlen*, long *n*, long *prec*)

void **acb_poly_upper_gamma_series** (acb_poly_t *res*, const acb_t *s*, const acb_poly_t *z*, long *n*, long *prec*)
Sets *res* to the upper incomplete gamma function $\Gamma(s, z)$ where *s* is a constant and *z* is a power series, truncated to length *n*.

void **_acb_poly_agm1_series** (acb_ptr *res*, acb_srcptr *z*, long *zlen*, long *len*, long *prec*)

void **acb_poly_agm1_series** (acb_poly_t *res*, const acb_poly_t *z*, long *n*, long *prec*)
Sets *res* to the arithmetic-geometric mean of 1 and the power series *z*, truncated to length *n*.

void **_acb_poly_elliptic_k_series** (acb_ptr *res*, acb_srcptr *z*, long *zlen*, long *len*, long *prec*)

void **acb_poly_elliptic_k_series** (acb_poly_t *res*, const acb_poly_t *z*, long *n*, long *prec*)
Sets *res* to the complete elliptic integral of the first kind of the power series *z*, truncated to length *n*.

void **_acb_poly_elliptic_p_series** (acb_ptr *res*, acb_srcptr *z*, long *zlen*, const acb_t *tau*, long *len*, long *prec*)

void **acb_poly_elliptic_p_series** (acb_poly_t *res*, const acb_poly_t *z*, const acb_t *tau*, long *n*, long *prec*)
Sets *res* to the Weierstrass elliptic function of the power series *z*, with periods 1 and *tau*, truncated to length *n*.

3.8.20 Root-finding

void **_acb_poly_root_inclusion** (acb_t *r*, const acb_t *m*, acb_srcptr *poly*, acb_srcptr *polyder*, long *len*, long *prec*)
Given any complex number m, and a nonconstant polynomial f and its derivative f', sets *r* to a complex interval centered on m that is guaranteed to contain at least one root of f. Such an interval is obtained by taking a ball of radius $|f(m)/f'(m)|n$ where n is the degree of f. Proof: assume that the distance to the nearest root exceeds $r = |f(m)/f'(m)|n$. Then

$$\left|\frac{f'(m)}{f(m)}\right| = \left|\sum_i \frac{1}{m-\zeta_i}\right| \le \sum_i \frac{1}{|m-\zeta_i|} < \frac{n}{r} = \left|\frac{f'(m)}{f(m)}\right|$$

which is a contradiction (see [Kob2010]).

long **_acb_poly_validate_roots** (acb_ptr *roots*, acb_srcptr *poly*, long *len*, long *prec*)

Given a list of approximate roots of the input polynomial, this function sets a rigorous bounding interval for each root, and determines which roots are isolated from all the other roots. It then rearranges the list of roots so that the isolated roots are at the front of the list, and returns the count of isolated roots.

If the return value equals the degree of the polynomial, then all roots have been found. If the return value is smaller, all the remaining output intervals are guaranteed to contain roots, but it is possible that not all of the polynomial's roots are contained among them.

void **_acb_poly_refine_roots_durand_kerner** (acb_ptr *roots*, acb_srcptr *poly*, long *len*, long *prec*)

Refines the given roots simultaneously using a single iteration of the Durand-Kerner method. The radius of each root is set to an approximation of the correction, giving a rough estimate of its error (not a rigorous bound).

long **_acb_poly_find_roots** (acb_ptr *roots*, acb_srcptr *poly*, acb_srcptr *initial*, long *len*, long *maxiter*, long *prec*)

long **acb_poly_find_roots** (acb_ptr *roots*, const acb_poly_t *poly*, acb_srcptr *initial*, long *maxiter*, long *prec*)

Attempts to compute all the roots of the given nonzero polynomial *poly* using a working precision of *prec* bits. If *n* denotes the degree of *poly*, the function writes *n* approximate roots with rigorous error bounds to the preallocated array *roots*, and returns the number of roots that are isolated.

If the return value equals the degree of the polynomial, then all roots have been found. If the return value is smaller, all the output intervals are guaranteed to contain roots, but it is possible that not all of the polynomial's roots are contained among them.

The roots are computed numerically by performing several steps with the Durand-Kerner method and terminating if the estimated accuracy of the roots approaches the working precision or if the number of steps exceeds *maxiter*, which can be set to zero in order to use a default value. Finally, the approximate roots are validated rigorously.

Initial values for the iteration can be provided as the array *initial*. If *initial* is set to *NULL*, default values $(0.4 + 0.9i)^k$ are used.

The polynomial is assumed to be squarefree. If there are repeated roots, the iteration is likely to find them (with low numerical accuracy), but the error bounds will not converge as the precision increases.

3.9 acb_mat.h – matrices over the complex numbers

An `acb_mat_t` represents a dense matrix over the complex numbers, implemented as an array of entries of type `acb_struct`.

The dimension (number of rows and columns) of a matrix is fixed at initialization, and the user must ensure that inputs and outputs to an operation have compatible dimensions. The number of rows or columns in a matrix can be zero.

3.9.1 Types, macros and constants

acb_mat_struct

acb_mat_t

Contains a pointer to a flat array of the entries (entries), an array of pointers to the start of each row (rows), and the number of rows (r) and columns (c).

An *acb_mat_t* is defined as an array of length one of type *acb_mat_struct*, permitting an *acb_mat_t* to be passed by reference.

acb_mat_entry (mat, i, j)

Macro giving a pointer to the entry at row *i* and column *j*.

acb_mat_nrows (mat)
Returns the number of rows of the matrix.

acb_mat_ncols (mat)
Returns the number of columns of the matrix.

3.9.2 Memory management

void **acb_mat_init** (acb_mat_t *mat*, long *r*, long *c*)
Initializes the matrix, setting it to the zero matrix with *r* rows and *c* columns.

void **acb_mat_clear** (acb_mat_t *mat*)
Clears the matrix, deallocating all entries.

3.9.3 Conversions

void **acb_mat_set** (acb_mat_t *dest*, const acb_mat_t *src*)

void **acb_mat_set_fmpz_mat** (acb_mat_t *dest*, const fmpz_mat_t *src*)

void **acb_mat_set_fmpq_mat** (acb_mat_t *dest*, const fmpq_mat_t *src*, long *prec*)
Sets *dest* to *src*. The operands must have identical dimensions.

3.9.4 Random generation

void **acb_mat_randtest** (acb_mat_t *mat*, flint_rand_t *state*, long *prec*, long *mag_bits*)
Sets *mat* to a random matrix with up to *prec* bits of precision and with exponents of width up to *mag_bits*.

3.9.5 Input and output

void **acb_mat_printd** (const acb_mat_t *mat*, long *digits*)
Prints each entry in the matrix with the specified number of decimal digits.

3.9.6 Comparisons

int **acb_mat_equal** (const acb_mat_t *mat1*, const acb_mat_t *mat2*)
Returns nonzero iff the matrices have the same dimensions and identical entries.

int **acb_mat_overlaps** (const acb_mat_t *mat1*, const acb_mat_t *mat2*)
Returns nonzero iff the matrices have the same dimensions and each entry in *mat1* overlaps with the corresponding entry in *mat2*.

int **acb_mat_contains** (const acb_mat_t *mat1*, const acb_mat_t *mat2*)

int **acb_mat_contains_fmpz_mat** (const acb_mat_t *mat1*, const fmpz_mat_t *mat2*)

int **acb_mat_contains_fmpq_mat** (const acb_mat_t *mat1*, const fmpq_mat_t *mat2*)
Returns nonzero iff the matrices have the same dimensions and each entry in *mat2* is contained in the corresponding entry in *mat1*.

int **acb_mat_is_real** (const acb_mat_t *mat*)
Returns nonzero iff all entries in *mat* have zero imaginary part.

3.9.7 Special matrices

void **acb_mat_zero** (acb_mat_t *mat*)
Sets all entries in mat to zero.

void **acb_mat_one** (acb_mat_t *mat*)
Sets the entries on the main diagonal to ones, and all other entries to zero.

3.9.8 Norms

void **acb_mat_bound_inf_norm** (mag_t *b*, const acb_mat_t *A*)
Sets *b* to an upper bound for the infinity norm (i.e. the largest absolute value row sum) of *A*.

3.9.9 Arithmetic

void **acb_mat_neg** (acb_mat_t *dest*, const acb_mat_t *src*)
Sets *dest* to the exact negation of *src*. The operands must have the same dimensions.

void **acb_mat_add** (acb_mat_t *res*, const acb_mat_t *mat1*, const acb_mat_t *mat2*, long *prec*)
Sets res to the sum of *mat1* and *mat2*. The operands must have the same dimensions.

void **acb_mat_sub** (acb_mat_t *res*, const acb_mat_t *mat1*, const acb_mat_t *mat2*, long *prec*)
Sets *res* to the difference of *mat1* and *mat2*. The operands must have the same dimensions.

void **acb_mat_mul** (acb_mat_t *res*, const acb_mat_t *mat1*, const acb_mat_t *mat2*, long *prec*)
Sets *res* to the matrix product of *mat1* and *mat2*. The operands must have compatible dimensions for matrix multiplication.

void **acb_mat_pow_ui** (acb_mat_t *res*, const acb_mat_t *mat*, ulong *exp*, long *prec*)
Sets *res* to *mat* raised to the power *exp*. Requires that *mat* is a square matrix.

3.9.10 Scalar arithmetic

void **acb_mat_scalar_mul_2exp_si** (acb_mat_t *B*, const acb_mat_t *A*, long *c*)
Sets *B* to *A* multiplied by 2^c.

void **acb_mat_scalar_addmul_si** (acb_mat_t *B*, const acb_mat_t *A*, long *c*, long *prec*)

void **acb_mat_scalar_addmul_fmpz** (acb_mat_t *B*, const acb_mat_t *A*, const fmpz_t *c*, long *prec*)

void **acb_mat_scalar_addmul_arb** (acb_mat_t *B*, const acb_mat_t *A*, const arb_t *c*, long *prec*)

void **acb_mat_scalar_addmul_acb** (acb_mat_t *B*, const acb_mat_t *A*, const acb_t *c*, long *prec*)
Sets *B* to $B + A \times c$.

void **acb_mat_scalar_mul_si** (acb_mat_t *B*, const acb_mat_t *A*, long *c*, long *prec*)

void **acb_mat_scalar_mul_fmpz** (acb_mat_t *B*, const acb_mat_t *A*, const fmpz_t *c*, long *prec*)

void **acb_mat_scalar_mul_arb** (acb_mat_t *B*, const acb_mat_t *A*, const arb_t *c*, long *prec*)

void **acb_mat_scalar_mul_acb** (acb_mat_t *B*, const acb_mat_t *A*, const acb_t *c*, long *prec*)
Sets *B* to $A \times c$.

void **acb_mat_scalar_div_si** (acb_mat_t *B*, const acb_mat_t *A*, long *c*, long *prec*)

void **acb_mat_scalar_div_fmpz** (acb_mat_t *B*, const acb_mat_t *A*, const fmpz_t *c*, long *prec*)

void **acb_mat_scalar_div_arb** (acb_mat_t *B*, const acb_mat_t *A*, const arb_t *c*, long *prec*)

void **acb_mat_scalar_div_acb** (acb_mat_t *B*, const acb_mat_t *A*, const acb_t *c*, long *prec*)
Sets *B* to A/c.

3.9.11 Gaussian elimination and solving

int **acb_mat_lu** (long * *perm*, acb_mat_t *LU*, const acb_mat_t *A*, long *prec*)
Given an $n \times n$ matrix A, computes an LU decomposition $PLU = A$ using Gaussian elimination with partial pivoting. The input and output matrices can be the same, performing the decomposition in-place.

Entry i in the permutation vector perm is set to the row index in the input matrix corresponding to row i in the output matrix.

The algorithm succeeds and returns nonzero if it can find n invertible (i.e. not containing zero) pivot entries. This guarantees that the matrix is invertible.

The algorithm fails and returns zero, leaving the entries in P and LU undefined, if it cannot find n invertible pivot elements. In this case, either the matrix is singular, the input matrix was computed to insufficient precision, or the LU decomposition was attempted at insufficient precision.

void **acb_mat_solve_lu_precomp** (acb_mat_t *X*, const long * *perm*, const acb_mat_t *LU*, const acb_mat_t *B*, long *prec*)
Solves $AX = B$ given the precomputed nonsingular LU decomposition $A = PLU$. The matrices X and B are allowed to be aliased with each other, but X is not allowed to be aliased with LU.

int **acb_mat_solve** (acb_mat_t *X*, const acb_mat_t *A*, const acb_mat_t *B*, long *prec*)
Solves $AX = B$ where A is a nonsingular $n \times n$ matrix and X and B are $n \times m$ matrices, using LU decomposition.

If $m > 0$ and A cannot be inverted numerically (indicating either that A is singular or that the precision is insufficient), the values in the output matrix are left undefined and zero is returned. A nonzero return value guarantees that A is invertible and that the exact solution matrix is contained in the output.

int **acb_mat_inv** (acb_mat_t *X*, const acb_mat_t *A*, long *prec*)
Sets $X = A^{-1}$ where A is a square matrix, computed by solving the system $AX = I$.

If A cannot be inverted numerically (indicating either that A is singular or that the precision is insufficient), the values in the output matrix are left undefined and zero is returned. A nonzero return value guarantees that the matrix is invertible and that the exact inverse is contained in the output.

void **acb_mat_det** (acb_t *det*, const acb_mat_t *A*, long *prec*)
Computes the determinant of the matrix, using Gaussian elimination with partial pivoting. If at some point an invertible pivot element cannot be found, the elimination is stopped and the magnitude of the determinant of the remaining submatrix is bounded using Hadamard's inequality.

3.9.12 Characteristic polynomial

void **_acb_mat_charpoly** (acb_ptr *cp*, const acb_mat_t *mat*, long *prec*)

void **acb_mat_charpoly** (acb_poly_t *cp*, const acb_mat_t *mat*, long *prec*)
Sets *cp* to the characteristic polynomial of *mat* which must be a square matrix. If the matrix has *n* rows, the underscore method requires space for $n + 1$ output coefficients. Employs a division-free algorithm using $O(n^4)$ operations.

3.9.13 Special functions

void **acb_mat_exp** (acb_mat_t *B*, const acb_mat_t *A*, long *prec*)

Sets B to the exponential of the matrix A, defined by the Taylor series

$$\exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

The function is evaluated as $\exp(A/2^r)^{2^r}$, where r is chosen to give rapid convergence of the Taylor series. The series is evaluated using rectangular splitting. If $\|A/2^r\| \le c$ and $N \ge 2c$, we bound the entrywise error when truncating the Taylor series before term N by $2c^N/N!$.

3.10 acb_calc.h – calculus with complex-valued functions

This module provides functions for operations of calculus over the complex numbers (intended to include root-finding, integration, and so on).

3.10.1 Types, macros and constants

acb_calc_func_t

Typedef for a pointer to a function with signature:

```
int func(acb_ptr out, const acb_t inp, void * param, long order, long prec)
```

implementing a univariate complex function $f(x)$. When called, *func* should write to *out* the first *order* coefficients in the Taylor series expansion of $f(x)$ at the point *inp*, evaluated at a precision of *prec* bits. The *param* argument may be used to pass through additional parameters to the function. The return value is reserved for future use as an error code. It can be assumed that *out* and *inp* are not aliased and that *order* is positive.

3.10.2 Bounds

void **acb_calc_cauchy_bound** (arb_t *bound*, acb_calc_func_t *func*, void * *param*, const acb_t *x*, const arb_t *radius*, long *maxdepth*, long *prec*)

Sets *bound* to a ball containing the value of the integral

$$C(x,r) = \frac{1}{2\pi r} \oint_{|z-x|=r} |f(z)| dz = \int_0^1 |f(x + re^{2\pi i t})| dt$$

where *f* is specified by (*func*, *param*) and *r* is given by *radius*. The integral is computed using a simple step sum. The integration range is subdivided until the order of magnitude of *b* can be determined (i.e. its error bound is smaller than its midpoint), or until the step length has been cut in half *maxdepth* times. This function is currently implemented completely naively, and repeatedly subdivides the whole integration range instead of performing adaptive subdivisions.

3.10.3 Integration

int **acb_calc_integrate_taylor** (acb_t *res*, acb_calc_func_t *func*, void * *param*, const acb_t *a*, const acb_t *b*, const arf_t *inner_radius*, const arf_t *outer_radius*, long *accuracy_goal*, long *prec*)

Computes the integral

$$I = \int_a^b f(t) dt$$

where f is specified by (*func*, *param*), following a straight-line path between the complex numbers a and b which both must be finite.

The integral is approximated by piecewise centered Taylor polynomials. Rigorous truncation error bounds are calculated using the Cauchy integral formula. More precisely, if the Taylor series of f centered at the point *m* is $f(m+x) = \sum_{n=0}^{\infty} a_n x^n$, then

$$\int f(m+x) = \left(\sum_{n=0}^{N-1} a_n \frac{x^{n+1}}{n+1} \right) + \left(\sum_{n=N}^{\infty} a_n \frac{x^{n+1}}{n+1} \right).$$

For sufficiently small *x*, the second series converges and its absolute value is bounded by

$$\sum_{n=N}^{\infty} \frac{C(m,R)}{R^n} \frac{|x|^{n+1}}{N+1} = \frac{C(m,R)Rx}{(R-x)(N+1)} \left(\frac{x}{R}\right)^N.$$

It is required that any singularities of f are isolated from the path of integration by a distance strictly greater than the positive value *outer_radius* (which is the integration radius used for the Cauchy bound). Taylor series step lengths are chosen so as not to exceed *inner_radius*, which must be strictly smaller than *outer_radius* for convergence. A smaller *inner_radius* gives more rapid convergence of each Taylor series but means that more series might have to be used. A reasonable choice might be to set *inner_radius* to half the value of *outer_radius*, giving roughly one accurate bit per term.

The truncation point of each Taylor series is chosen so that the absolute truncation error is roughly 2^{-p} where *p* is given by *accuracy_goal* (in the future, this might change to a relative accuracy). Arithmetic operations and function evaluations are performed at a precision of *prec* bits. Note that due to accumulation of numerical errors, both values may have to be set higher (and the endpoints may have to be computed more accurately) to achieve a desired accuracy.

This function chooses the evaluation points uniformly rather than implementing adaptive subdivision.

3.11 acb_hypgeom.h – hypergeometric functions in the complex numbers

The generalized hypergeometric function is formally defined by

$${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{z^k}{k!}.$$

It can be interpreted using analytic continuation or regularization when the sum does not converge. In a looser sense, we understand “hypergeometric functions” to be linear combinations of generalized hypergeometric functions with prefactors that are products of exponentials, powers, and gamma functions.

3.11.1 Convergent series

In this section, we define

$$T(k) = \frac{\prod_{i=0}^{p-1} (a_i)_k}{\prod_{i=0}^{q-1} (b_i)_k} z^k$$

and

$${}_pH_q(a_0,\ldots,a_{p-1};b_0\ldots b_{q-1};z) = {}_{p+1}F_q(a_0,\ldots,a_{p-1},1;b_0\ldots b_{q-1};z) = \sum_{k=0}^{\infty} T(k)$$

For the conventional generalized hypergeometric function ${}_pF_q$, compute ${}_pH_{q+1}$ with the explicit parameter $b_q = 1$, or remove a 1 from the a_i parameters if there is one.

void **acb_hypgeom_pfq_bound_factor** (mag_t *C*, acb_srcptr *a*, long *p*, acb_srcptr *b*, long *q*, const acb_t *z*, ulong *n*)

Computes a factor *C* such that

$$\left|\sum_{k=n}^{\infty} T(k)\right| \le C|T(n)|.$$

We check that $\operatorname{Re}(b+n) > 0$ for all lower parameters *b*. If this does not hold, *C* is set to infinity. Otherwise, we cancel out pairs of parameters a and b against each other. We have

$$\left|\frac{a+k}{b+k}\right| = \left|1 + \frac{a-b}{b+k}\right| \le 1 + \frac{|a-b|}{|b+n|}$$

and

$$\left|\frac{1}{b+k}\right| \le \frac{1}{|b+n|}$$

for all $k \ge n$. This gives us a constant *D* such that $T(k+1) \le DT(k)$ for all $k \ge n$. If $D \ge 1$, we set *C* to infinity. Otherwise, we take $C = \sum_{k=0}^{\infty} D^k = (1-D)^{-1}$.

As currently implemented, the bound becomes infinite when n is too small, even if the series converges.

long **acb_hypgeom_pfq_choose_n** (acb_srcptr *a*, long *p*, acb_srcptr *b*, long *q*, const acb_t *z*, long *prec*)

Heuristically attempts to choose a number of terms *n* to sum of a hypergeometric series at a working precision of *prec* bits.

Uses double precision arithmetic internally. As currently implemented, it can fail to produce a good result if the parameters are extremely large or extremely close to nonpositive integers.

Numerical cancellation is assumed to be significant, so truncation is done when the current term is *prec* bits smaller than the largest encountered term.

This function will also attempt to pick a reasonable truncation point for divergent series.

void **acb_hypgeom_pfq_sum_forward** (acb_t *s*, acb_t *t*, acb_srcptr *a*, long *p*, acb_srcptr *b*, long *q*, const acb_t *z*, long *n*, long *prec*)

void **acb_hypgeom_pfq_sum_rs** (acb_t *s*, acb_t *t*, acb_srcptr *a*, long *p*, acb_srcptr *b*, long *q*, const acb_t *z*, long *n*, long *prec*)

void **acb_hypgeom_pfq_sum** (acb_t *s*, acb_t *t*, acb_srcptr *a*, long *p*, acb_srcptr *b*, long *q*, const acb_t *z*, long *n*, long *prec*)

Computes $s = \sum_{k=0}^{n-1} T(k)$ and $t = T(n)$. Does not allow aliasing between input and output variables. We require $n \ge 0$.

The *forward* version computes the sum using forward recurrence.

The *rs* version computes the sum in reverse order using rectangular splitting. It only computes a magnitude bound for the value of *t*.

The default version automatically chooses an algorithm depending on the inputs.

void **acb_hypgeom_pfq_direct** (acb_t *res*, acb_srcptr *a*, long *p*, acb_srcptr *b*, long *q*, const acb_t *z*, long *n*, long *prec*)

Computes

$${}_pH_q(z) = \sum_{k=0}^{\infty} T(k) = \sum_{k=0}^{n-1} T(k) + \varepsilon$$

directly from the defining series, including a rigorous bound for the truncation error ε in the output.

If $n < 0$, this function chooses a number of terms automatically using `acb_hypgeom_pfq_choose_n()`.

3.11.2 Asymptotic series

Let $U(a,b,z)$ denote the confluent hypergeometric function of the second kind with the principal branch cut, and let $U^* = z^a U(a,b,z)$. For all $z \neq 0$ and $b \notin \mathbb{Z}$ (but valid for all b as a limit), we have (DLMF 13.2.42)

$$U(a,b,z) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)} M(a,b,z) + \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} M(a-b+1, 2-b, z).$$

Moreover, for all $z \neq 0$ we have

$$\frac{{}_1F_1(a,b,z)}{\Gamma(b)} = \frac{(-z)^{-a}}{\Gamma(b-a)} U^*(a,b,z) + \frac{z^{a-b} e^z}{\Gamma(a)} U^*(b-a, b, -z)$$

which is equivalent to DLMF 13.2.41 (but simpler in form).

We have the asymptotic expansion

$$U^*(a,b,z) \sim {}_2F_0(a, a-b+1, -1/z)$$

where ${}_2F_0(a,b,z)$ denotes a formal hypergeometric series, i.e.

$$U^*(a,b,z) = \sum_{k=0}^{n-1} \frac{(a)_k (a-b+1)_k}{k!(-z)^k} + \varepsilon_n(z).$$

The error term $\varepsilon_n(z)$ is bounded according to DLMF 13.7. A case distinction is made depending on whether z lies in one of three regions which we index by R. Our formula for the error bound increases with the value of R, so we can always choose the larger out of two indices if z lies in the union of two regions.

Let $r = |b - 2a|$. If $\operatorname{Re}(z) \geq r$, set $R = 1$. Otherwise, if $\operatorname{Im}(z) \geq r$ or $\operatorname{Re}(z) \geq 0 \wedge |z| \geq r$, set $R = 2$. Otherwise, if $|z| \geq 2r$, set $R = 3$. Otherwise, the bound is infinite. If the bound is finite, we have

$$|\varepsilon_n(z)| \leq 2\alpha C_n \left| \frac{(a)_n (a-b+1)_n}{n! z^n} \right| \exp(2\alpha\rho C_1/|z|)$$

in terms of the following auxiliary quantities

$$\sigma = |(b-2a)/z|$$

$$C_n = \begin{cases} 1 & \text{if } R = 1 \\ \chi(n) & \text{if } R = 2 \\ (\chi(n) + \rho\nu^2 n)\nu^n & \text{if } R = 3 \end{cases}$$

$$\nu = \left(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1 - 4\sigma^2}\right)^{-1/2} \leq 1 + 2\sigma^2$$

$$\chi(n) = \sqrt{\pi}\,\Gamma(\tfrac{1}{2}n + 1)/\Gamma(\tfrac{1}{2}n + \tfrac{1}{2})$$

$$\sigma' = \begin{cases} \sigma & \text{if } R \neq 3 \\ \nu\sigma & \text{if } R = 3 \end{cases}$$

$$\alpha = (1 - \sigma')^{-1}$$

$$\rho = \tfrac{1}{2}|2a^2 - 2ab + b| + \sigma'(1 + \tfrac{1}{4}\sigma')(1 - \sigma')^{-2}$$

void **acb_hypgeom_u_asymp** (acb_t *res*, const acb_t *a*, const acb_t *b*, const acb_t *z*, long *n*, long *prec*)

Sets *res* to $U^*(a,b,z)$ computed using n terms of the asymptotic series, with a rigorous bound for the error included in the output. We require $n \geq 0$.

3.11.3 The error function

void **acb_hypgeom_erf_1f1a** (acb_t *res*, const acb_t *z*, long *prec*)

void **acb_hypgeom_erf_1f1b** (acb_t *res*, const acb_t *z*, long *prec*)

void **acb_hypgeom_erf_asymp** (acb_t *res*, const acb_t *z*, long *prec*, long *prec2*)

void **acb_hypgeom_erf** (acb_t *res*, const acb_t *z*, long *prec*)

Computes the error function respectively using

$$\operatorname{erf}(z) = \frac{2z}{\sqrt{\pi}} {}_1F_1(\tfrac{1}{2}, \tfrac{3}{2}, -z^2)$$

$$\operatorname{erf}(z) = \frac{2ze^{-z^2}}{\sqrt{\pi}} {}_1F_1(1, \tfrac{3}{2}, z^2)$$

$$\operatorname{erf}(z) = \frac{z}{\sqrt{z^2}} \left(1 - \frac{e^{-z^2}}{\sqrt{\pi}} U(\tfrac{1}{2}, \tfrac{1}{2}, z^2)\right).$$

and an automatic algorithm choice. The *asymp* version takes a second precision to use for the U term.

void **acb_hypgeom_erfc** (acb_t *res*, const acb_t *s*, const acb_t *z*, long *prec*)

Computes the complementary error function $\operatorname{erfc}(z) = 1 - \operatorname{erf}(z)$. This function avoids catastrophic cancellation for large positive z.

void **acb_hypgeom_erfi** (acb_t *res*, const acb_t *s*, const acb_t *z*, long *prec*)

Computes the imaginary error function $\operatorname{erfi}(z) = -i \operatorname{erf}(iz)$. This is a trivial wrapper of `acb_hypgeom_erf()`.

3.11.4 Bessel functions

void **acb_hypgeom_bessel_j_asymp** (acb_t *res*, const acb_t *nu*, const acb_t *z*, long *prec*)

Computes the Bessel function of the first kind via `acb_hypgeom_u_asymp()`. For all complex ν, z, we have

$$J_\nu(z) = \frac{z^\nu}{2^\nu e^{iz} \Gamma(\nu + 1)} {}_1F_1(\nu + \tfrac{1}{2}, 2\nu + 1, 2iz) = A_+ B_+ + A_- B_-$$

where

$$A_\pm = z^\nu (z^2)^{-\frac{1}{2}-\nu} (\mp iz)^{\frac{1}{2}+\nu} (2\pi)^{-1/2} = (\pm iz)^{-1/2-\nu} z^\nu (2\pi)^{-1/2}$$

$$B_\pm = e^{\pm iz} U^*(\nu + \tfrac{1}{2}, 2\nu + 1, \mp 2iz).$$

Nicer representations of the factors $A_\pm$ can be given depending conditionally on the parameters. If $\nu + \frac{1}{2} = n \in \mathbb{Z}$, we have $A_\pm = (\pm i)^n (2\pi z)^{-1/2}$. And if $\operatorname{Re}(z) > 0$, we have $A_\pm = \exp(\mp i[(2\nu + 1)/4]\pi)(2\pi z)^{-1/2}$.

void **acb_hypgeom_bessel_j_0f1** (acb_t *res*, const acb_t *nu*, const acb_t *z*, long *prec*)

Computes the Bessel function of the first kind from

$$J_\nu(z) = \frac{1}{\Gamma(\nu + 1)} \left(\frac{z}{2}\right)^\nu {}_0F_1\left(\nu + 1, -\frac{z^2}{4}\right).$$

void **acb_hypgeom_bessel_j** (acb_t *res*, const acb_t *nu*, const acb_t *z*, long *prec*)

Computes the Bessel function of the first kind $J_\nu(z)$ using an automatic algorithm choice.

3.11.5 Incomplete gamma functions

void **acb_hypgeom_gamma_upper_asymp** (acb_t *res*, const acb_t *s*, const acb_t *z*, int *modified*, long *prec*)

void **acb_hypgeom_gamma_upper_1f1a** (acb_t *res*, const acb_t *s*, const acb_t *z*, int *modified*, long *prec*)

void **acb_hypgeom_gamma_upper_1f1b** (acb_t *res*, const acb_t *s*, const acb_t *z*, int *modified*, long *prec*)

void **acb_hypgeom_gamma_upper_singular** (acb_t *res*, long *s*, const acb_t *z*, int *modified*, long *prec*)

void **acb_hypgeom_gamma_upper** (acb_t *res*, const acb_t *s*, const acb_t *z*, int *modified*, long *prec*)

Computes the upper incomplete gamma function respectively using

$$\Gamma(s,z) = e^{-z} U(1-s, 1-s, z)$$

$$\Gamma(s,z) = \Gamma(s) - \frac{z^s}{s} {}_1F_1(s, s+1, -z)$$

$$\Gamma(s,z) = \Gamma(s) - \frac{z^s e^{-z}}{s} {}_1F_1(1, s+1, z)$$

$$\Gamma(s,z) = \frac{(-1)^n}{n!} (\psi(n+1) - \log(z)) + \frac{(-1)^n}{(n+1)!} z \, {}_2F_2(1,1,2,2+n,-z) - z^{-n} \sum_{k=0}^{n-1} \frac{(-z)^k}{(k-n)k!}, \quad n = -s \in \mathbb{Z}_{\geq 0}$$

and an automatic algorithm choice. The automatic version also handles other special input such as $z = 0$ and $s = 1, 2, 3$. The *singular* version evaluates the finite sum directly and therefore assumes that *s* is not too large. If *modified* is set, computes the exponential integral $z^{-s}\Gamma(s,z) = E_{1-s}(z)$ instead.

void **acb_hypgeom_expint** (acb_t *res*, const acb_t *s*, const acb_t *z*, long *prec*)

Computes the exponential integral $E_s(z)$. This is a trivial wrapper of `acb_hypgeom_gamma_upper()`.

3.12 acb_modular.h – modular forms in the complex numbers

This module provides methods for numerical evaluation of modular forms, Jacobi theta functions, and elliptic functions.

In the context of this module, *tau* or τ always denotes an element of the complex upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$. We also often use the variable q, variously defined as $q = e^{2\pi i \tau}$ (usually in relation to modular forms) or $q = e^{\pi i \tau}$ (usually in relation to theta functions) and satisfying $|q| < 1$. We will clarify the local meaning of q every time such a quantity appears as a function of τ.

As usual, the numerical functions in this module compute strict error bounds: if *tau* is represented by an `acb_t` whose content overlaps with the real line (or lies in the lower half-plane), and *tau* is passed to a function defined only on $\mathbb{H}$, then the output will have an infinite radius. The analogous behavior holds for functions requiring $|q| < 1$.

3.12.1 The modular group

psl2z_struct

psl2z_t

Represents an element of the modular group $\mathrm{PSL}(2,\mathbb{Z})$, namely an integer matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $ad - bc = 1$, and with signs canonicalized such that $c \geq 0$, and $d > 0$ if $c = 0$. The struct members *a*, *b*, *c*, *d* are of type *fmpz*.

void **psl2z_init** (psl2z_t *g*)
    Initializes *g* and set it to the identity element.

void **psl2z_clear** (psl2z_t *g*)
    Clears *g*.

void **psl2z_swap** (psl2z_t *f*, psl2z_t *g*)
    Swaps *f* and *g* efficiently.

void **psl2z_set** (psl2z_t *f*, const psl2z_t *g*)
    Sets *f* to a copy of *g*.

void **psl2z_one** (psl2z_t *g*)
    Sets *g* to the identity element.

int **psl2z_is_one** (const psl2z_t *g*)
    Returns nonzero iff *g* is the identity element.

void **psl2z_print** (const psl2z_t *g*)
    Prints *g* to standard output.

int **psl2z_equal** (const psl2z_t *f*, const psl2z_t *g*)
    Returns nonzero iff *f* and *g* are equal.

void **psl2z_mul** (psl2z_t *h*, const psl2z_t *f*, const psl2z_t *g*)
    Sets *h* to the product of *f* and *g*, namely the matrix product with the signs canonicalized.

void **psl2z_inv** (psl2z_t *h*, const psl2z_t *g*)
    Sets *h* to the inverse of *g*.

int **psl2z_is_correct** (const psl2z_t *g*)
    Returns nonzero iff *g* contains correct data, i.e. satisfying $ad - bc = 1$, $c \geq 0$, and $d > 0$ if $c = 0$.

void **psl2z_randtest** (psl2z_t *g*, flint_rand_t *state*, long *bits*)
    Sets *g* to a random element of $\text{PSL}(2, \mathbb{Z})$ with entries of bit length at most *bits* (or 1, if *bits* is not positive). We first generate *a* and *d*, compute their Bezout coefficients, divide by the GCD, and then correct the signs.

3.12.2 Modular transformations

void **acb_modular_transform** (acb_t *w*, const psl2z_t *g*, const acb_t *z*, long *prec*)
    Applies the modular transformation *g* to the complex number *z*, evaluating

$$w = gz = \frac{az + b}{cz + d}.$$

void **acb_modular_fundamental_domain_approx_d** (psl2z_t *g*, double *x*, double *y*, double *one_minus_eps*)

void **acb_modular_fundamental_domain_approx_arf** (psl2z_t *g*, const arf_t *x*, const arf_t *y*, const arf_t *one_minus_eps*, long *prec*)
    Attempts to determine a modular transformation *g* that maps the complex number $x + yi$ to the fundamental domain or just slightly outside the fundamental domain, where the target tolerance (not a strict bound) is specified by *one_minus_eps*.

    The inputs are assumed to be finite numbers, with *y* positive.

    Uses floating-point iteration, repeatedly applying either the transformation $z \leftarrow z + b$ or $z \leftarrow -1/z$. The iteration is terminated if $|x| \leq 1/2$ and $x^2 + y^2 \geq 1 - \varepsilon$ where $1 - \varepsilon$ is passed as *one_minus_eps*. It is also terminated if too many steps have been taken without convergence, or if the numbers end up too large or too small for the working precision.

The algorithm can fail to produce a satisfactory transformation. The output g is always set to *some* correct modular transformation, but it is up to the user to verify a posteriori that g maps $x + yi$ close enough to the fundamental domain.

void **acb_modular_fundamental_domain_approx** (acb_t *w*, psl2z_t *g*, const acb_t *z*, const arf_t *one_minus_eps*, long *prec*)

Attempts to determine a modular transformation g that maps the complex number z to the fundamental domain or just slightly outside the fundamental domain, where the target tolerance (not a strict bound) is specified by *one_minus_eps*. It also computes the transformed value $w = gz$.

This function first tries to use `acb_modular_fundamental_domain_approx_d()` and checks if the result is acceptable. If this fails, it calls `acb_modular_fundamental_domain_approx_arf()` with higher precision. Finally, $w = gz$ is evaluated by a single application of g.

The algorithm can fail to produce a satisfactory transformation. The output g is always set to *some* correct modular transformation, but it is up to the user to verify a posteriori that w is close enough to the fundamental domain.

int **acb_modular_is_in_fundamental_domain** (const acb_t *z*, const arf_t *tol*, long *prec*)

Returns nonzero if it is certainly true that $|z| \geq 1 - \varepsilon$ and $|\operatorname{Re}(z)| \leq 1/2 + \varepsilon$ where ε is specified by *tol*. Returns zero if this is false or cannot be determined.

3.12.3 Jacobi theta functions

Unfortunately, there are many inconsistent notational variations for Jacobi theta functions in the literature. Unless otherwise noted, we use the functions

$$\theta_1(z,\tau) = -i \sum_{n=-\infty}^{\infty} (-1)^n \exp(\pi i [(n+1/2)^2 \tau + (2n+1)z]) = 2 q_{1/4} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \sin((2n+1)\pi z)$$

$$\theta_2(z,\tau) = \sum_{n=-\infty}^{\infty} \exp(\pi i [(n+1/2)^2 \tau + (2n+1)z]) = 2 q_{1/4} \sum_{n=0}^{\infty} q^{n(n+1)} \cos((2n+1)\pi z)$$

$$\theta_3(z,\tau) = \sum_{n=-\infty}^{\infty} \exp(\pi i [n^2 \tau + 2nz]) = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos(2n\pi z)$$

$$\theta_4(z,\tau) = \sum_{n=-\infty}^{\infty} (-1)^n \exp(\pi i [n^2 \tau + 2nz]) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos(2n\pi z)$$

where $q = \exp(\pi i \tau)$ and $q_{1/4} = \exp(\pi i \tau / 4)$. Note that many authors write $q_{1/4}$ as $q^{1/4}$, but the principal fourth root $(q)^{1/4} = \exp(\frac{1}{4} \log q)$ differs from $q_{1/4}$ in general and some formulas are only correct if one reads "$q^{1/4} = \exp(\pi i \tau / 4)$". To avoid confusion, we only write q^k when k is an integer.

void **acb_modular_theta_transform** (int * *R*, int * *S*, int * *C*, const psl2z_t *g*)

We wish to write a theta function with quasiperiod τ in terms of a theta function with quasiperiod $\tau' = g\tau$, given some $g = (a, b; c, d) \in \mathrm{PSL}(2, \mathbb{Z})$. For $i = 0, 1, 2, 3$, this function computes integers R_i and S_i (*R* and *S* should be arrays of length 4) and $C \in \{0, 1\}$ such that

$$\theta_{1+i}(z,\tau) = \exp(\pi i R_i / 4) \cdot A \cdot B \cdot \theta_{1+S_i}(z', \tau')$$

where $z' = z$, $A = B = 1$ if $C = 0$, and

$$z' = \frac{-z}{c\tau + d}, \quad A = \sqrt{\frac{i}{c\tau + d}}, \quad B = \exp\left(-\pi i c \frac{z^2}{c\tau + d}\right)$$

if $C = 1$. Note that A is well-defined with the principal branch of the square root since $A^2 = i/(c\tau + d)$ lies in the right half-plane.

Firstly, if $c = 0$, we have $\theta_i(z, \tau) = \exp(-\pi ib/4)\theta_i(z, \tau + b)$ for $i = 1, 2$, whereas θ_3 and θ_4 remain unchanged when b is even and swap places with each other when b is odd. In this case we set $C = 0$.

For an arbitrary g with $c > 0$, we set $C = 1$. The general transformations are given by Rademacher [Rad1973]. We need the function $\theta_{m,n}(z, \tau)$ defined for $m, n \in \mathbb{Z}$ by (beware of the typos in [Rad1973])

$$\theta_{0,0}(z,\tau) = \theta_3(z,\tau), \quad \theta_{0,1}(z,\tau) = \theta_4(z,\tau)$$
$$\theta_{1,0}(z,\tau) = \theta_2(z,\tau), \quad \theta_{1,1}(z,\tau) = i\theta_1(z,\tau)$$
$$\theta_{m+2,n}(z,\tau) = (-1)^n \theta_{m,n}(z,\tau)$$
$$\theta_{m,n+2}(z,\tau) = \theta_{m,n}(z,\tau).$$

Then we may write

$$\theta_1(z,\tau) = \varepsilon_1 AB\theta_1(z',\tau')$$
$$\theta_2(z,\tau) = \varepsilon_2 AB\theta_{1-c,1+a}(z',\tau')$$
$$\theta_3(z,\tau) = \varepsilon_3 AB\theta_{1+d-c,1-b+a}(z',\tau')$$
$$\theta_4(z,\tau) = \varepsilon_4 AB\theta_{1+d,1-b}(z',\tau')$$

where ε_i is an 8th root of unity. Specifically, if we denote the 24th root of unity in the transformation formula of the Dedekind eta function by $\varepsilon(a, b, c, d) = \exp(\pi iR(a, b, c, d)/12)$ (see `acb_modular_epsilon_arg()`), then:

$$\varepsilon_1(a,b,c,d) = \exp(\pi i[R(-d,b,c,-a)+1]/4)$$
$$\varepsilon_2(a,b,c,d) = \exp(\pi i[-R(a,b,c,d)+(5+(2-c)a)]/4)$$
$$\varepsilon_3(a,b,c,d) = \exp(\pi i[-R(a,b,c,d)+(4+(c-d-2)(b-a))]/4)$$
$$\varepsilon_4(a,b,c,d) = \exp(\pi i[-R(a,b,c,d)+(3-(2+d)b)]/4)$$

These formulas are easily derived from the formulas in [Rad1973] (Rademacher has the transformed/untransformed variables exchanged, and his "ε" differs from ours by a constant offset in the phase).

void **acb_modular_addseq_theta**(long * *exponents*, long * *aindex*, long * *bindex*, long *num*)
Constructs an addition sequence for the first *num* squares and triangular numbers interleaved (excluding zero), i.e. 1, 2, 4, 6, 9, 12, 16, 20, 25, 30 etc.

void **acb_modular_theta_sum**(acb_ptr *theta1*, acb_ptr *theta2*, acb_ptr *theta3*, acb_ptr *theta4*, const acb_t *w*, int *w_is_unit*, const acb_t *q*, long *len*, long *prec*)
Simultaneously computes the first *len* coefficients of each of the formal power series

$$\theta_1(z+x,\tau)/q_{1/4} \in \mathbb{C}[[x]]$$
$$\theta_2(z+x,\tau)/q_{1/4} \in \mathbb{C}[[x]]$$
$$\theta_3(z+x,\tau) \in \mathbb{C}[[x]]$$
$$\theta_4(z+x,\tau) \in \mathbb{C}[[x]]$$

given $w = \exp(\pi iz)$ and $q = \exp(\pi i\tau)$, by summing a finite truncation of the respective theta function series. In particular, with *len* equal to 1, computes the respective value of the theta function at the point z. We require *len* to be positive. If *w_is_unit* is nonzero, *w* is assumed to lie on the unit circle, i.e. z is assumed to be real.

Note that the factor $q_{1/4}$ is removed from θ_1 and θ_2. To get the true theta function values, the user has to multiply this factor back. This convention avoids unnecessary computations, since the user can compute $q_{1/4} = \exp(\pi i\tau/4)$ followed by $q = (q_{1/4})^4$, and in many cases when computing products or quotients of theta functions, the factor $q_{1/4}$ can be eliminated entirely.

This function is intended for $|q| \ll 1$. It can be called with any q, but will return useless intervals if convergence is not rapid. For general evaluation of theta functions, the user should only call this function after applying a suitable modular transformation.

We consider the sums together, alternatingly updating (θ_1, θ_2) or (θ_3, θ_4). For $k = 0, 1, 2, \ldots$, the powers of q are $\lfloor (k+2)^2/4 \rfloor = 1, 2, 4, 6, 9$ etc. and the powers of w are $\pm(k+2) = \pm 2, \pm 3, \pm 4, \ldots$ etc. The scheme is illustrated by the following table:

$$\begin{array}{llll} & \theta_1, \theta_2 & q^0 & (w^1 \pm w^{-1}) \\ k=0 & \theta_3, \theta_4 & q^1 & (w^2 \pm w^{-2}) \\ k=1 & \theta_1, \theta_2 & q^2 & (w^3 \pm w^{-3}) \\ k=2 & \theta_3, \theta_4 & q^4 & (w^4 \pm w^{-4}) \\ k=3 & \theta_1, \theta_2 & q^6 & (w^5 \pm w^{-5}) \\ k=4 & \theta_3, \theta_4 & q^9 & (w^6 \pm w^{-6}) \\ k=5 & \theta_1, \theta_2 & q^{12} & (w^7 \pm w^{-7}) \end{array}$$

For some integer $N \geq 1$, the summation is stopped just before term $k = N$. Let $Q = |q|$, $W = \max(|w|, |w^{-1}|)$, $E = \lfloor (N+2)^2/4 \rfloor$ and $F = \lfloor (N+1)/2 \rfloor + 1$. The error of the zeroth derivative can be bounded as

$$2Q^E W^{N+2} \left[1 + Q^F W + Q^{2F} W^2 + \ldots\right] = \frac{2Q^E W^{N+2}}{1 - Q^F W}$$

provided that the denominator is positive (otherwise we set the error bound to infinity). When *len* is greater than 1, consider the derivative of order *r*. The term of index *k* and order *r* picks up a factor of magnitude $(k+2)^r$ from differentiation of w^{k+2} (it also picks up a factor π^r, but we omit this until we rescale the coefficients at the end of the computation). Thus we have the error bound

$$2Q^E W^{N+2} (N+2)^r \left[1 + Q^F W \frac{(N+3)^r}{(N+2)^r} + Q^{2F} W^2 \frac{(N+4)^r}{(N+2)^r} + \ldots\right]$$

which by the inequality $(1 + m/(N+2))^r \le \exp(mr/(N+2))$ can be bounded as

$$\frac{2Q^E W^{N+2} (N+2)^r}{1 - Q^F W \exp(r/(N+2))},$$

again valid when the denominator is positive.

To actually evaluate the series, we write the even cosine terms as $w^{2n} + w^{-2n}$, the odd cosine terms as $w(w^{2n} + w^{-2n-2})$, and the sine terms as $w(w^{2n} - w^{-2n-2})$. This way we only need even powers of w and w^{-1}. The implementation is not yet optimized for real z, in which case further work can be saved.

This function does not permit aliasing between input and output arguments.

void **acb_modular_theta_notransform** (acb_t *theta1*, acb_t *theta2*, acb_t *theta3*, acb_t *theta4*, const acb_t *z*, const acb_t *tau*, long *prec*)

Evaluates the Jacobi theta functions $\theta_i(z, \tau)$, $i = 1, 2, 3, 4$ simultaneously. This function does not move τ to the fundamental domain. This is generally worse than `acb_modular_theta()`, but can be slightly better for moderate input.

void **acb_modular_theta** (acb_t *theta1*, acb_t *theta2*, acb_t *theta3*, acb_t *theta4*, const acb_t *z*, const acb_t *tau*, long *prec*)

Evaluates the Jacobi theta functions $\theta_i(z, \tau)$, $i = 1, 2, 3, 4$ simultaneously. This function moves τ to the fundamental domain before calling `acb_modular_theta_sum()`.

3.12.4 The Dedekind eta function

void **acb_modular_addseq_eta** (long * *exponents*, long * *aindex*, long * *bindex*, long *num*)

Constructs an addition sequence for the first *num* generalized pentagonal numbers (excluding zero), i.e. 1, 2, 5, 7, 12, 15, 22, 26, 35, 40 etc.

void **acb_modular_eta_sum** (acb_t *eta*, const acb_t *q*, long *prec*)
Evaluates the Dedekind eta function without the leading 24th root, i.e.

$$\exp(-\pi i\tau/12)\eta(\tau) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(3n^2-n)/2}$$

given $q = \exp(2\pi i\tau)$, by summing the defining series.

This function is intended for $|q| \ll 1$. It can be called with any q, but will return useless intervals if convergence is not rapid. For general evaluation of the eta function, the user should only call this function after applying a suitable modular transformation.

int **acb_modular_epsilon_arg** (const psl2z_t *g*)
Given $g = (a, b; c, d)$, computes an integer R such that $\varepsilon(a, b, c, d) = \exp(\pi iR/12)$ is the 24th root of unity in the transformation formula for the Dedekind eta function,

$$\eta\left(\frac{a\tau + b}{c\tau + d}\right) = \varepsilon(a, b, c, d)\sqrt{c\tau + d}\eta(\tau).$$

void **acb_modular_eta** (acb_t *r*, const acb_t *tau*, long *prec*)
Computes the Dedekind eta function $\eta(\tau)$ given τ in the upper half-plane. This function applies the functional equation to move τ to the fundamental domain before calling `acb_modular_eta_sum()`.

3.12.5 Modular forms

void **acb_modular_j** (acb_t *r*, const acb_t *tau*, long *prec*)
Computes Klein's j-invariant $j(\tau)$ given τ in the upper half-plane. The function is normalized so that $j(i) = 1728$. We first move τ to the fundamental domain, which does not change the value of the function. Then we use the formula $j(\tau) = 32(\theta_2^8 + \theta_3^8 + \theta_4^8)^3/(\theta_2\theta_3\theta_4)^8$ where $\theta_i = \theta_i(0, \tau)$.

void **acb_modular_lambda** (acb_t *r*, const acb_t *tau*, long *prec*)
Computes the lambda function $\lambda(\tau) = \theta_2^4(0, \tau)/\theta_3^4(0, \tau)$, which is invariant under modular transformations $(a, b; c, d)$ where a, d are odd and b, c are even.

void **acb_modular_delta** (acb_t *r*, const acb_t *tau*, long *prec*)
Computes the modular discriminant $\Delta(\tau) = \eta(\tau)^{24}$, which transforms as

$$\Delta\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{12}\Delta(\tau).$$

The modular discriminant is sometimes defined with an extra factor $(2\pi)^{12}$, which we omit in this implementation.

void **acb_modular_eisenstein** (acb_ptr *r*, const acb_t *tau*, long *len*, long *prec*)
Computes simultaneously the first *len* entries in the sequence of Eisenstein series $G_4(\tau), G_6(\tau), G_8(\tau), \ldots$, defined by

$$G_{2k}(\tau) = \sum_{m^2+n^2\neq 0} \frac{1}{(m + n\tau)^{2k}}$$

and satisfying

$$G_{2k}\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{2k}G_{2k}(\tau).$$

We first evaluate $G_4(\tau)$ and $G_6(\tau)$ on the fundamental domain using theta functions, and then compute the Eisenstein series of higher index using a recurrence relation.

3.12.6 Elliptic functions

void **acb_modular_elliptic_p**(acb_t *wp*, const acb_t *z*, const acb_t *tau*, long *prec*)

Computes Weierstrass's elliptic function

$$\wp(z,\tau) = \frac{1}{z^2} + \sum_{n^2+m^2 \ne 0} \left[\frac{1}{(z+m+n\tau)^2} - \frac{1}{(m+n\tau)^2} \right]$$

which satisfies $\wp(z,\tau) = \wp(z+1,\tau) = \wp(z+\tau,\tau)$. To evaluate the function efficiently, we use the formula

$$\wp(z,\tau) = \pi^2 \theta_2^2(0,\tau)\theta_3^2(0,\tau)\frac{\theta_4^2(z,\tau)}{\theta_1^2(z,\tau)} - \frac{\pi^2}{3}\left[\theta_3^4(0,\tau) + \theta_3^4(0,\tau)\right].$$

void **acb_modular_elliptic_p_zpx**(acb_ptr *wp*, const acb_t *z*, const acb_t *tau*, long *len*, long *prec*)

Computes the formal power series $\wp(z+x,\tau) \in \mathbb{C}[[x]]$, truncated to length *len*. In particular, with *len* = 2, simultaneously computes $\wp(z,\tau), \wp'(z,\tau)$ which together generate the field of elliptic functions with periods 1 and τ.

3.12.7 Elliptic integrals

void **acb_modular_elliptic_k**(acb_t *w*, const acb_t *m*, long *prec*)

Computes the complete elliptic integral of the first kind $K(m)$, using the arithmetic-geometric mean: $K(m) = \pi/(2M(\sqrt{1-m}))$.

void **acb_modular_elliptic_k_cpx**(acb_ptr *w*, const acb_t *m*, long *len*, long *prec*)

Sets the coefficients in the array *w* to the power series expansion of the complete elliptic integral of the first kind at the point *m* truncated to length *len*, i.e. $K(m+x) \in \mathbb{C}[[x]]$.

3.13 bernoulli.h – support for Bernoulli numbers

This module provides helper functions for exact or approximate calculation of the Bernoulli numbers, which are defined by the exponential generating function

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.$$

Efficient algorithms are implemented for both multi-evaluation and calculation of isolated Bernoulli numbers. A global (or thread-local) cache is also provided, to support fast repeated evaluation of various special functions that depend on the Bernoulli numbers (including the gamma function and the Riemann zeta function).

3.13.1 Generation of Bernoulli numbers

bernoulli_rev_t

An iterator object for generating a range of even-indexed Bernoulli numbers exactly in reverse order, i.e. computing the exact fractions $B_n, B_{n-2}, B_{n-4}, \ldots, B_0$. The Bernoulli numbers are generated from scratch, i.e. no caching is performed.

The Bernoulli numbers are computed by direct summation of the zeta series. This is made fast by storing a table of powers (as done by Bloemen et al. http://remcobloemen.nl/2009/11/even-faster-zeta-calculation.html). As an optimization, we only include the odd powers, and use fixed-point arithmetic.

The reverse iteration order is preferred for performance reasons, as the powers can be updated using multiplications instead of divisions, and we avoid having to periodically recompute terms to higher precision. To generate

Bernoulli numbers in the forward direction without having to store all of them, one can split the desired range into smaller blocks and compute each block with a single reverse pass.

void **bernoulli_rev_init** (bernoulli_rev_t *iter*, ulong *n*)
Initializes the iterator *iter*. The first Bernoulli number to be generated by calling `bernoulli_rev_next()` is B_n. It is assumed that n is even.

void **bernoulli_rev_next** (fmpz_t *numer*, fmpz_t *denom*, bernoulli_rev_t *iter*)
Sets *numer* and *denom* to the exact, reduced numerator and denominator of the Bernoulli number B_k and advances the state of *iter* so that the next invocation generates B_{k-2}.

void **bernoulli_rev_clear** (bernoulli_rev_t *iter*)
Frees all memory allocated internally by *iter*.

3.13.2 Caching

long **bernoulli_cache_num**

fmpq * **bernoulli_cache**
Cache of Bernoulli numbers. Uses thread-local storage if enabled in FLINT.

void **bernoulli_cache_compute** (long *n*)
Makes sure that the Bernoulli numbers up to at least B_{n-1} are cached. Calling `flint_cleanup()` frees the cache.

3.13.3 Bounding

long **bernoulli_bound_2exp_si** (ulong *n*)
Returns an integer b such that $|B_n| \le 2^b$. Uses a lookup table for small n, and for larger n uses the inequality $|B_n| < 4n!/(2\pi)^n < 4(n+1)^{n+1}e^{-n}/(2\pi)^n$. Uses integer arithmetic throughout, with the bound for the logarithm being looked up from a table. If $|B_n| = 0$, returns *LONG_MIN*. Otherwise, the returned exponent b is never more than one percent larger than the true magnitude.

This function is intended for use when n small enough that one might comfortably compute B_n exactly. It aborts if n is so large that internal overflow occurs.

void **_bernoulli_fmpq_ui_zeta** (fmpz_t *num*, fmpz_t *den*, ulong *n*)
Sets *num* and *den* to the reduced numerator and denominator of the Bernoulli number B_n.

This function computes the denominator d using von Staudt-Clausen theorem, numerically approximates B_n using `arb_bernoulli_ui_zeta()`, and then rounds dB_n to the correct numerator. If the working precision is insufficient to determine the numerator, the function prints a warning message and retries with increased precision (this should not be expected to happen).

void **_bernoulli_fmpq_ui** (fmpz_t *num*, fmpz_t *den*, ulong *n*)

void **bernoulli_fmpq_ui** (fmpq_t *b*, ulong *n*)
Computes the Bernoulli number B_n as an exact fraction, for an isolated integer n. This function reads B_n from the global cache if the number is already cached, but does not automatically extend the cache by itself.

3.14 hypgeom.h – support for hypergeometric series

This module provides functions for high-precision evaluation of series of the form

$$\sum_{k=0}^{n-1} \frac{A(k)}{B(k)} \prod_{j=1}^{k} \frac{P(k)}{Q(k)} z^k$$

where A, B, P, Q are polynomials. The present version only supports $A, B, P, Q \in \mathbb{Z}[k]$ (represented using the FLINT *fmpz_poly_t* type). This module also provides functions for high-precision evaluation of infinite series ($n \to \infty$), with automatic, rigorous error bounding.

Note that we can standardize to $A = B = 1$ by setting $\tilde{P}(k) = P(k)A(k)B(k-1), \tilde{Q}(k) = Q(k)A(k-1)B(k)$. However, separating out A and B is convenient and improves efficiency during evaluation.

3.14.1 Strategy for error bounding

We wish to evaluate $S(z) = \sum_{k=0}^{\infty} T(k)z^k$ where $T(k)$ satisfies $T(0) = 1$ and

$$T(k) = R(k)T(k-1) = \left(\frac{P(k)}{Q(k)}\right) T(k-1)$$

for given polynomials

$$P(k) = a_p k^p + a_{p-1} k^{p-1} + \ldots a_0$$
$$Q(k) = b_q k^q + b_{q-1} k^{q-1} + \ldots b_0.$$

For convergence, we require $p < q$, or $p = q$ with $|z||a_p| < |b_q|$. We also assume that $P(k)$ and $Q(k)$ have no roots among the positive integers (if there are positive integer roots, the sum is either finite or undefined). With these conditions satisfied, our goal is to find a parameter $n \geq 0$ such that

$$\left| \sum_{k=n}^{\infty} T(k)z^k \right| \leq 2^{-d}.$$

We can rewrite the hypergeometric term ratio as

$$zR(k) = z\frac{P(k)}{Q(k)} = z\left(\frac{a_p}{b_q}\right)\frac{1}{k^{q-p}}F(k)$$

where

$$F(k) = \frac{1 + \tilde{a}_1/k + \tilde{a}_2/k^2 + \ldots + \tilde{a}_q/k^p}{1 + \tilde{b}_1/k + \tilde{b}_2/k^2 + \ldots + \tilde{b}_q/k^q} = 1 + O(1/k)$$

and where $\tilde{a}_i = a_{p-i}/a_p$, $\tilde{b}_i = b_{q-i}/b_q$. Next, we define

$$C = \max_{1 \leq i \leq p} |\tilde{a}_i|^{(1/i)}, \quad D = \max_{1 \leq i \leq q} |\tilde{b}_i|^{(1/i)}.$$

Now, if $k > C$, the magnitude of the numerator of $F(k)$ is bounded from above by

$$1 + \sum_{i=1}^{p} \left(\frac{C}{k}\right)^i \leq 1 + \frac{C}{k-C}$$

and if $k > 2D$, the magnitude of the denominator of $F(k)$ is bounded from below by

$$1 - \sum_{i=1}^{q} \left(\frac{D}{k}\right)^i \geq 1 + \frac{D}{D-k}.$$

Putting the inequalities together gives the following bound, valid for $k > K = \max(C, 2D)$:

$$|F(k)| \le \frac{k(k-D)}{(k-C)(k-2D)} = \left(1 + \frac{C}{k-C}\right)\left(1 + \frac{D}{k-2D}\right).$$

Let $r = q - p$ and $\tilde{z} = |z a_p / b_q|$. Assuming $k > \max(C, 2D, \tilde{z}^{1/r})$, we have

$$|zR(k)| \le G(k) = \frac{\tilde{z} F(k)}{k^r}$$

where $G(k)$ is monotonically decreasing. Now we just need to find an n such that $G(n) < 1$ and for which $|T(n)|/(1 - G(n)) \le 2^{-d}$. This can be done by computing a floating-point guess for n then trying successively larger values.

This strategy leaves room for some improvement. For example, if $\tilde{b}_1$ is positive and large, the bound B becomes very pessimistic (a larger positive $\tilde{b}_1$ causes faster convergence, not slower convergence).

3.14.2 Types, macros and constants

hypgeom_struct

hypgeom_t

Stores polynomials *A*, *B*, *P*, *Q* and precomputed bounds, representing a fixed hypergeometric series.

3.14.3 Memory management

void **hypgeom_init** (hypgeom_t *hyp*)

void **hypgeom_clear** (hypgeom_t *hyp*)

3.14.4 Error bounding

long **hypgeom_estimate_terms** (const mag_t *z*, int *r*, long *d*)

Computes an approximation of the largest n such that $|z|^n/(n!)^r = 2^{-d}$, giving a first-order estimate of the number of terms needed to approximate the sum of a hypergeometric series of weight $r \ge 0$ and argument z to an absolute precision of $d \ge 0$ bits. If $r = 0$, the direct solution of the equation is given by $n = (\log(1-z) - d\log 2)/\log z$. If $r > 0$, using $\log n! \approx n \log n - n$ gives an equation that can be solved in terms of the Lambert *W*-function as $n = (d \log 2)/(r\, W(t))$ where $t = (d \log 2)/(e r z^{1/r})$.

The evaluation is done using double precision arithmetic. The function aborts if the computed value of n is greater than or equal to LONG_MAX / 2.

long **hypgeom_bound** (mag_t *error*, int *r*, long *C*, long *D*, long *K*, const mag_t *TK*, const mag_t *z*, long *prec*)

Computes a truncation parameter sufficient to achieve *prec* bits of absolute accuracy, according to the strategy described above. The input consists of r, C, D, K, precomputed bound for $T(K)$, and $\tilde{z} = z(a_p/b_q)$, such that for $k > K$, the hypergeometric term ratio is bounded by

$$\frac{\tilde{z}}{k^r} \frac{k(k-D)}{(k-C)(k-2D)}.$$

Given this information, we compute a ε and an integer n such that $|\sum_{k=n}^{\infty} T(k)| \le \varepsilon \le 2^{-\mathrm{prec}}$. The output variable *error* is set to the value of ε, and n is returned.

void **hypgeom_precompute** (hypgeom_t *hyp*)

Precomputes the bounds data C, D, K and an upper bound for $T(K)$.

3.14.5 Summation

void **fmprb_hypgeom_sum**(fmprb_t *P*, fmprb_t *Q*, const hypgeom_t *hyp*, const long *n*, long *prec*)
Computes P, Q such that $P/Q = \sum_{k=0}^{n-1} T(k)$ where $T(k)$ is defined by *hyp*, using binary splitting and a working precision of *prec* bits.

void **fmprb_hypgeom_infsum**(fmprb_t *P*, fmprb_t *Q*, hypgeom_t *hyp*, long *tol*, long *prec*)
Computes P, Q such that $P/Q = \sum_{k=0}^{\infty} T(k)$ where $T(k)$ is defined by *hyp*, using binary splitting and working precision of *prec* bits. The number of terms is chosen automatically to bound the truncation error by at most $2^{-\mathrm{tol}}$. The bound for the truncation error is included in the output as part of *P*.

void **arb_hypgeom_sum**(arb_t *P*, arb_t *Q*, const hypgeom_t *hyp*, const long *n*, long *prec*)
Computes P, Q such that $P/Q = \sum_{k=0}^{n-1} T(k)$ where $T(k)$ is defined by *hyp*, using binary splitting and a working precision of *prec* bits.

void **arb_hypgeom_infsum**(arb_t *P*, arb_t *Q*, hypgeom_t *hyp*, long *tol*, long *prec*)
Computes P, Q such that $P/Q = \sum_{k=0}^{\infty} T(k)$ where $T(k)$ is defined by *hyp*, using binary splitting and working precision of *prec* bits. The number of terms is chosen automatically to bound the truncation error by at most $2^{-\mathrm{tol}}$. The bound for the truncation error is included in the output as part of *P*.

3.15 partitions.h – computation of the partition function

This module implements the asymptotically fast algorithm for evaluating the integer partition function $p(n)$ described in [Joh2012]. The idea is to evaluate a truncation of the Hardy-Ramanujan-Rademacher series using tight precision estimates, and symbolically factoring the occurring exponential sums.

An implementation based on floating-point arithmetic can also be found in FLINT. That version relies on some numerical subroutines that have not been proved correct.

The implementation provided here uses ball arithmetic throughout to guarantee a correct error bound for the numerical approximation of $p(n)$. Optionally, hardware double arithmetic can be used for low-precision terms. This gives a significant speedup for small (e.g. $n < 10^6$).

void **partitions_rademacher_bound**(arf_t *b*, const fmpz_t *n*, ulong *N*)
Sets b to an upper bound for

$$M(n, N) = \frac{44\pi^2}{225\sqrt{3}} N^{-1/2} + \frac{\pi\sqrt{2}}{75} \left(\frac{N}{n-1}\right)^{1/2} \sinh\left(\frac{\pi}{N}\sqrt{\frac{2n}{3}}\right).$$

This formula gives an upper bound for the truncation error in the Hardy-Ramanujan-Rademacher formula when the series is taken up to the term $t(n, N)$ inclusive.

partitions_hrr_sum_arb(arb_t *x*, const fmpz_t *n*, long *N0*, long *N*, int *use_doubles*)
Evaluates the partial sum $\sum_{k=N_0}^{N} t(n, k)$ of the Hardy-Ramanujan-Rademacher series.

If *use_doubles* is nonzero, doubles and the system's standard library math functions are used to evaluate the smallest terms. This significantly speeds up evaluation for small n (e.g. $n < 10^6$), and gives a small speed improvement for larger n, but the result is not guaranteed to be correct. In practice, the error is estimated very conservatively, and unless the system's standard library is broken, use of doubles can be considered safe. Setting *use_doubles* to zero gives a fully guaranteed bound.

void **partitions_fmpz_fmpz**(fmpz_t *p*, const fmpz_t *n*, int *use_doubles*)
Computes the partition function $p(n)$ using the Hardy-Ramanujan-Rademacher formula. This function computes a numerical ball containing $p(n)$ and verifies that the ball contains a unique integer.

If *n* is sufficiently large and a number of threads greater than 1 has been selected with `flint_set_num_threads()`, the computation time will be reduced by using two threads.

See `partitions_hrr_sum_arb()` for an explanation of the *use_doubles* option.

void **partitions_fmpz_ui** (fmpz_t *p*, ulong *n*)

Computes the partition function $p(n)$ using the Hardy-Ramanujan-Rademacher formula. This function computes a numerical ball containing $p(n)$ and verifies that the ball contains a unique integer.

void **partitions_fmpz_ui_using_doubles** (fmpz_t *p*, ulong *n*)

Computes the partition function $p(n)$, enabling the use of doubles internally. This significantly speeds up evaluation for small n (e.g. $n < 10^6$), but the error bounds are not certified (see remarks for `partitions_hrr_sum_arb()`).

CHAPTER

FOUR

ALGORITHMS AND PROOFS

4.1 Algorithms for mathematical constants

Most mathematical constants are evaluated using the generic hypergeometric summation code.

4.1.1 Pi

π is computed using the Chudnovsky series

$$\frac{1}{\pi} = 12 \sum_{k=0}^{\infty} \frac{(-1)^k (6k)! (13591409 + 545140134k)}{(3k)!(k!)^3 640320^{3k+3/2}}$$

which is hypergeometric and adds roughly 14 digits per term. Methods based on the arithmetic-geometric mean seem to be slower by a factor three in practice.

A small trick is to compute $1/\sqrt{640320}$ instead of $\sqrt{640320}$ at the end.

4.1.2 Logarithms of integers

We use the formulas

$$\log(2) = \frac{3}{4} \sum_{k=0}^{\infty} \frac{(-1)^k (k!)^2}{2^k (2k+1)!}$$

$$\log(10) = 46 \operatorname{atanh}(1/31) + 34 \operatorname{atanh}(1/49) + 20 \operatorname{atanh}(1/161)$$

4.1.3 Euler's constant

Euler's constant γ is computed using the Brent-McMillan formula ([BM1980], [MPFR2012])

$$\gamma = \frac{S_0(2n) - K_0(2n)}{I_0(2n)} - \log(n)$$

in which n is a free parameter and

$$S_0(x) = \sum_{k=0}^{\infty} \frac{H_k}{(k!)^2} \left(\frac{x}{2}\right)^{2k}, \quad I_0(x) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{x}{2}\right)^{2k}$$

$$2x I_0(x) K_0(x) \sim \sum_{k=0}^{\infty} \frac{[(2k)!]^3}{(k!)^4 8^{2k} x^{2k}}.$$

All series are evaluated using binary splitting. The first two series are evaluated simultaneously, with the summation taken up to $k = N - 1$ inclusive where $N \ge \alpha n + 1$ and $\alpha \approx 4.970625759544231864 4$ satisfies $\alpha(\log \alpha - 1) = 3$. The third series is taken up to $k = 2n - 1$ inclusive. With these parameters, it is shown in [BJ2013] that the error is bounded by $24e^{-8n}$.

4.1.4 Catalan's constant

Catalan's constant is computed using the hypergeometric series

$$C = \sum_{k=0}^{\infty} \frac{(-1)^k 4^{4k+1} \left(40k^2 + 56k + 19\right) [(k+1)!]^2 [(2k+2)!]^3}{(k+1)^3 (2k+1) [(4k+4)!]^2}$$

4.1.5 Khinchin's constant

Khinchin's constant K_0 is computed using the formula

$$\log K_0 = \frac{1}{\log 2} \left[\sum_{k=2}^{N-1} \log\left(\frac{k-1}{k}\right) \log\left(\frac{k+1}{k}\right) + \sum_{n=1}^{\infty} \frac{\zeta(2n, N)}{n} \sum_{k=1}^{2n-1} \frac{(-1)^{k+1}}{k} \right]$$

where $N \ge 2$ is a free parameter that can be used for tuning [BBC1997]. If the infinite series is truncated after $n = M$, the remainder is smaller in absolute value than

$$\sum_{n=M+1}^{\infty} \zeta(2n, N) = \sum_{n=M+1}^{\infty} \sum_{k=0}^{\infty} (k+N)^{-2n} \le \sum_{n=M+1}^{\infty} \left(N^{-2n} + \int_0^{\infty} (t+N)^{-2n} dt \right)$$
$$= \sum_{n=M+1}^{\infty} \frac{1}{N^{2n}} \left(1 + \frac{N}{2n-1}\right) \le \sum_{n=M+1}^{\infty} \frac{N+1}{N^{2n}} = \frac{1}{N^{2M}(N-1)} \le \frac{1}{N^{2M}}.$$

Thus, for an error of at most 2^{-p} in the series, it is sufficient to choose $M \ge p/(2 \log_2 N)$.

4.1.6 Glaisher's constant

Glaisher's constant $A = \exp(1/12 - \zeta'(-1))$ is computed directly from this formula. We don't use the reflection formula for the zeta function, as the arithmetic in Euler-Maclaurin summation is faster at $s = -1$ than at $s = 2$.

4.1.7 Apery's constant

Apery's constant $\zeta(3)$ is computed using the hypergeometric series

$$\zeta(3) = \frac{1}{64} \sum_{k=0}^{\infty} (-1)^k (205k^2 + 250k + 77) \frac{(k!)^{10}}{[(2k+1)!]^5}.$$

4.2 Algorithms for gamma functions

4.2.1 The Stirling series

In general, the gamma function is computed via the Stirling series

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right) \log z - z + \frac{\ln 2\pi}{2} + \sum_{k=1}^{n-1} \frac{B_{2k}}{2k(2k-1)z^{2k-1}} + R(n, z)$$

where ([Olv1997] pp. 293-295) the remainder term is exactly

$$R_n(z) = \int_0^\infty \frac{B_{2n} - \tilde{B}_{2n}(x)}{2n(x+z)^{2n}} dx.$$

To evaluate the gamma function of a power series argument, we substitute $z \to z + t \in \mathbb{C}[[t]]$.

Using the bound for $|x+z|$ given by [Olv1997] and the fact that the numerator of the integrand is bounded in absolute value by $2|B_{2n}|$, the remainder can be shown to satisfy the bound

$$|[t^k]R_n(z+t)| \le 2|B_{2n}| \frac{\Gamma(2n+k-1)}{\Gamma(k+1)\Gamma(2n+1)} \; |z| \; (b/|z|)^{2n+k}$$

where $b = 1/\cos(\arg(z)/2)$. Note that by trigonometric identities, assuming that $z = x + yi$, we have $b = \sqrt{1+t^2}$ where

$$t = \frac{y}{\sqrt{x^2+y^2}+x} = \frac{\sqrt{x^2+y^2}-x}{y}.$$

To use the Stirling series at p-bit precision, we select parameters r, n such that the remainder $R(n,z)$ approximately is bounded by 2^{-p}. If $|z|$ is too small for the Stirling series to give sufficient accuracy directly, we first translate to $z + r$ using the formula $\Gamma(z) = \Gamma(z+r)/(z(z+1)(z+2)\cdots(z+r-1))$.

To obtain a remainder smaller than 2^{-p}, we must choose an r such that, in the real case, $z + r > \beta p$, where $\beta > \log(2)/(2\pi) \approx 0.11$. In practice, a slightly larger factor $\beta \approx 0.2$ more closely balances n and r. A much larger β (e.g. $\beta = 1$) could be used to reduce the number of Bernoulli numbers that have to be precomputed, at the expense of slower repeated evaluation.

4.2.2 Rational arguments

We use efficient methods to compute $y = \Gamma(p/q)$ where q is one of $1, 2, 3, 4, 6$ and p is a small integer.

The cases $\Gamma(1) = 1$ and $\Gamma(1/2) = \sqrt{\pi}$ are trivial. We reduce all remaining cases to $\Gamma(1/3)$ or $\Gamma(1/4)$ using the following relations:

$$\Gamma(2/3) = \frac{2\pi}{3^{1/2}\Gamma(1/3)}, \qquad \Gamma(3/4) = \frac{2^{1/2}\pi}{\Gamma(1/4)},$$
$$\Gamma(1/6) = \frac{\Gamma(1/3)^2}{(\pi/3)^{1/2}2^{1/3}}, \qquad \Gamma(5/6) = \frac{2\pi(\pi/3)^{1/2}2^{1/3}}{\Gamma(1/3)^2}.$$

We compute $\Gamma(1/3)$ and $\Gamma(1/4)$ rapidly to high precision using

$$\Gamma(1/3) = \left(\frac{12\pi^4}{\sqrt{10}} \sum_{k=0}^{\infty} \frac{(6k)!(-1)^k}{(k!)^3(3k)!3^k160^{3k}}\right)^{1/6}, \qquad \Gamma(1/4) = \sqrt{\frac{(2\pi)^{3/2}}{\operatorname{agm}(1,\sqrt{2})}}.$$

An alternative formula which could be used for $\Gamma(1/3)$ is

$$\Gamma(1/3) = \frac{2^{4/9}\pi^{2/3}}{3^{1/12}\left(\operatorname{agm}\left(1, \frac{1}{2}\sqrt{2+\sqrt{3}}\right)\right)^{1/3}},$$

but this appears to be slightly slower in practice.

4.3 Algorithms for polylogarithms

The polylogarithm is defined for $s, z \in \mathbb{C}$ with $|z| < 1$ by

$$\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}$$

and for $|z| \geq 1$ by analytic continuation, except for the singular point $z = 1$.

4.3.1 Computation for small z

The power sum converges rapidly when $|z| \ll 1$. To compute the series expansion with respect to s, we substitute $s \to s + x \in \mathbb{C}[[x]]$ and obtain

$$\mathrm{Li}_{s+x}(z) = \sum_{d=0}^{\infty} x^d \frac{(-1)^d}{d!} \sum_{k=1}^{\infty} T(k)$$

where

$$T(k) = \frac{z^k \log^d(k)}{k^s}.$$

The remainder term $|\sum_{k=N}^{\infty} T(k)|$ is bounded via `mag_polylog_tail()`.

4.3.2 Expansion for general z

For general complex s, z, we write the polylogarithm as a sum of two Hurwitz zeta functions

$$\mathrm{Li}_s(z) = \frac{\Gamma(v)}{(2\pi)^v} \left[i^v \zeta\left(v, \frac{1}{2} + \frac{\log(-z)}{2\pi i}\right) + i^{-v} \zeta\left(v, \frac{1}{2} - \frac{\log(-z)}{2\pi i}\right) \right]$$

in which $s = 1 - v$. With the principal branch of $\log(-z)$, we obtain the conventional analytic continuation of the polylogarithm with a branch cut on $z \in (1, +\infty)$.

To compute the series expansion with respect to v, we substitute $v \to v + x \in \mathbb{C}[[x]]$ in this formula (at the end of the computation, we map $x \to -x$ to obtain the power series for $\mathrm{Li}_{s+x}(z)$). The right hand side becomes

$$\Gamma(v + x)[E_1 Z_1 + E_2 Z_2]$$

where $E_1 = (i/(2\pi))^{v+x}$, $Z_1 = \zeta(v + x, \ldots)$, $E_2 = (1/(2\pi i))^{v+x}$, $Z_2 = \zeta(v + x, \ldots)$.

When $v = 1$, the Z_1 and Z_2 terms become Laurent series with a leading $1/x$ term. In this case, we compute the deflated series $\tilde{Z}_1, \tilde{Z}_2 = \zeta(x, \ldots) - 1/x$. Then

$$E_1 Z_1 + E_2 Z_2 = (E_1 + E_2)/x + E_1 \tilde{Z}_1 + E_2 \tilde{Z}_2.$$

Note that $(E_1 + E_2)/x$ is a power series, since the constant term in $E_1 + E_2$ is zero when $v = 1$. So we simply compute one extra derivative of both E_1 and E_2, and shift them one step. When $v = 0, -1, -2, \ldots$, the $\Gamma(v + x)$ prefactor has a pole. In this case, we proceed analogously and formally multiply $x\,\Gamma(v + x)$ with $[E_1 Z_1 + E_2 Z_2]/x$.

Note that the formal cancellation only works when the order s (or v) is an exact integer: it is not currently possible to use this method when s is a small ball containing any of $0, 1, 2, \ldots$ (then the result becomes indeterminate).

The Hurwitz zeta method becomes inefficient when $|z| \to 0$ (it gives an indeterminate result when $z = 0$). This is not a problem since we just use the defining series for the polylogarithm in that region. It also becomes inefficient when $|z| \to \infty$, for which an asymptotic expansion would better.

CHAPTER FIVE

MODULE DOCUMENTATION (ARB 1.X TYPES)

5.1 fmpr.h – arbitrary-precision floating-point numbers

This type is now obsolete: use `arf_t` instead.

A variable of type `fmpr_t` holds an arbitrary-precision binary floating-point number, i.e. a rational number of the form $x \times 2^y$ where $x, y \in \mathbb{Z}$ and x is odd; or one of the special values zero, plus infinity, minus infinity, or NaN (not-a-number).

The component x is called the *mantissa*, and y is called the *exponent*. Note that this is just one among many possible conventions: the mantissa (alternatively *significand*) is sometimes viewed as a fraction in the interval $[1/2, 1)$, with the exponent pointing to the position above the top bit rather than the position of the bottom bit, and with a separate sign.

The conventions for special values largely follow those of the IEEE floating-point standard. At the moment, there is no support for negative zero, unsigned infinity, or a NaN with a payload, though some these might be added in the future.

An *fmpr* number is exact and has no inherent "accuracy". We use the term *precision* to denote either the target precision of an operation, or the bit size of a mantissa (which in general is unrelated to the "accuracy" of the number: for example, the floating-point value 1 has a precision of 1 bit in this sense and is simultaneously an infinitely accurate approximation of the integer 1 and a 2-bit accurate approximation of $\sqrt{2} = 1.011010100\ldots_2$).

Except where otherwise noted, the output of an operation is the floating-point number obtained by taking the inputs as exact numbers, in principle carrying out the operation exactly, and rounding the resulting real number to the nearest representable floating-point number whose mantissa has at most the specified number of bits, in the specified direction of rounding. Some operations are always or optionally done exactly.

5.1.1 Types, macros and constants

fmpr_struct

An *fmpr_struct* holds a mantissa and an exponent. If the mantissa and exponent are sufficiently small, their values are stored as immediate values in the *fmpr_struct*; large values are represented by pointers to heap-allocated arbitrary-precision integers. Currently, both the mantissa and exponent are implemented using the FLINT *fmpz* type. Special values are currently encoded by the mantissa being set to zero.

fmpr_t

An *fmpr_t* is defined as an array of length one of type *fmpr_struct*, permitting an *fmpr_t* to be passed by reference.

fmpr_rnd_t

Specifies the rounding mode for the result of an approximate operation.

FMPR_RND_DOWN
Specifies that the result of an operation should be rounded to the nearest representable number in the direction towards zero.

FMPR_RND_UP
Specifies that the result of an operation should be rounded to the nearest representable number in the direction away from zero.

FMPR_RND_FLOOR
Specifies that the result of an operation should be rounded to the nearest representable number in the direction towards minus infinity.

FMPR_RND_CEIL
Specifies that the result of an operation should be rounded to the nearest representable number in the direction towards plus infinity.

FMPR_RND_NEAR
Specifies that the result of an operation should be rounded to the nearest representable number, rounding to an odd mantissa if there is a tie between two values. *Warning*: this rounding mode is currently not implemented (except for a few conversions functions where this stated explicitly).

FMPR_PREC_EXACT
If passed as the precision parameter to a function, indicates that no rounding is to be performed. This must only be used when it is known that the result of the operation can be represented exactly and fits in memory (the typical use case is working small integer values). Note that, for example, adding two numbers whose exponents are far apart can easily produce an exact result that is far too large to store in memory.

5.1.2 Memory management

void **fmpr_init** (fmpr_t *x*)
Initializes the variable *x* for use. Its value is set to zero.

void **fmpr_clear** (fmpr_t *x*)
Clears the variable *x*, freeing or recycling its allocated memory.

5.1.3 Special values

void **fmpr_zero** (fmpr_t *x*)

void **fmpr_one** (fmpr_t *x*)

void **fmpr_pos_inf** (fmpr_t *x*)

void **fmpr_neg_inf** (fmpr_t *x*)

void **fmpr_nan** (fmpr_t *x*)
Sets *x* respectively to 0, 1, $+\infty$, $-\infty$, NaN.

int **fmpr_is_zero** (const fmpr_t *x*)

int **fmpr_is_one** (const fmpr_t *x*)

int **fmpr_is_pos_inf** (const fmpr_t *x*)

int **fmpr_is_neg_inf** (const fmpr_t *x*)

int **fmpr_is_nan** (const fmpr_t *x*)
Returns nonzero iff *x* respectively equals 0, 1, $+\infty$, $-\infty$, NaN.

int **fmpr_is_inf** (const fmpr_t *x*)
Returns nonzero iff *x* equals either $+\infty$ or $-\infty$.

int **fmpr_is_normal** (const fmpr_t *x*)
Returns nonzero iff *x* is a finite, nonzero floating-point value, i.e. not one of the special values 0, $+\infty$, $-\infty$, NaN.

int **fmpr_is_special** (const fmpr_t *x*)
Returns nonzero iff *x* is one of the special values 0, $+\infty$, $-\infty$, NaN, i.e. not a finite, nonzero floating-point value.

int **fmpr_is_finite** (fmpr_t *x*)
Returns nonzero iff *x* is a finite floating-point value, i.e. not one of the values $+\infty$, $-\infty$, NaN. (Note that this is not equivalent to the negation of `fmpr_is_inf()`.)

5.1.4 Assignment, rounding and conversions

long **_fmpr_normalise** (fmpz_t *man*, fmpz_t *exp*, long *prec*, fmpr_rnd_t *rnd*)
Rounds the mantissa and exponent in-place.

void **fmpr_set** (fmpr_t *y*, const fmpr_t *x*)
Sets *y* to a copy of *x*.

void **fmpr_swap** (fmpr_t *x*, fmpr_t *y*)
Swaps *x* and *y* efficiently.

long **fmpr_set_round** (fmpr_t *y*, const fmpr_t *x*, long *prec*, fmpr_rnd_t *rnd*)

long **fmpr_set_round_fmpz** (fmpr_t *x*, const fmpz_t *x*, long *prec*, fmpr_rnd_t *rnd*)
Sets *y* to a copy of *x* rounded in the direction specified by rnd to the number of bits specified by prec.

long **_fmpr_set_round_mpn** (long * *shift*, fmpz_t *man*, mp_srcptr *x*, mp_size_t *xn*, int *negative*, long *prec*, fmpr_rnd_t *rnd*)
Given an integer represented by a pointer *x* to a raw array of *xn* limbs (negated if *negative* is nonzero), sets *man* to the corresponding floating-point mantissa rounded to *prec* bits in direction *rnd*, sets *shift* to the exponent, and returns the error bound. We require that *xn* is positive and that the leading limb of *x* is nonzero.

long **fmpr_set_round_ui_2exp_fmpz** (fmpr_t *z*, mp_limb_t *lo*, const fmpz_t *exp*, int *negative*, long *prec*, fmpr_rnd_t *rnd*)
Sets *z* to the unsigned integer *lo* times two to the power *exp*, negating the value if *negative* is nonzero, and rounding the result to *prec* bits in direction *rnd*.

long **fmpr_set_round_uiui_2exp_fmpz** (fmpr_t *z*, mp_limb_t *hi*, mp_limb_t *lo*, const fmpz_t *exp*, int *negative*, long *prec*, fmpr_rnd_t *rnd*)
Sets *z* to the unsigned two-limb integer *{hi, lo}* times two to the power *exp*, negating the value if *negative* is nonzero, and rounding the result to *prec* bits in direction *rnd*.

void **fmpr_set_error_result** (fmpr_t *err*, const fmpr_t *result*, long *rret*)
Given the return value *rret* and output variable *result* from a function performing a rounding (e.g. *fmpr_set_round* or *fmpr_add*), sets *err* to a bound for the absolute error.

void **fmpr_add_error_result** (fmpr_t *err*, const fmpr_t *err_in*, const fmpr_t *result*, long *rret*, long *prec*, fmpr_rnd_t *rnd*)
Like *fmpr_set_error_result*, but adds *err_in* to the error.

void **fmpr_ulp** (fmpr_t *u*, const fmpr_t *x*, long *prec*)
Sets *u* to the floating-point unit in the last place (ulp) of *x*. The ulp is defined as in the MPFR documentation and satisfies $2^{-n}|x| < u \leq 2^{-n+1}|x|$ for any finite nonzero *x*. If *x* is a special value, *u* is set to the absolute value of *x*.

int **fmpr_check_ulp** (const fmpr_t *x*, long *r*, long *prec*)
Assume that *r* is the return code and *x* is the floating-point result from a single floating-point rounding. Then this function returns nonzero iff *x* and *r* define an error of exactly 0 or 1 ulp. In other words, this function checks that `fmpr_set_error_result()` gives exactly 0 or 1 ulp as expected.

int **fmpr_get_mpfr** (mpfr_t *x*, const fmpr_t *y*, mpfr_rnd_t *rnd*)
Sets the MPFR variable *x* to the value of *y*. If the precision of *x* is too small to allow *y* to be represented exactly, it is rounded in the specified MPFR rounding mode. The return value indicates the direction of rounding, following the standard convention of the MPFR library.

void **fmpr_set_mpfr** (fmpr_t *x*, const mpfr_t *y*)
Sets *x* to the exact value of the MPFR variable *y*.

double **fmpr_get_d** (const fmpr_t *x*, fmpr_rnd_t *rnd*)
Returns *x* rounded to a *double* in the direction specified by *rnd*.

void **fmpr_set_d** (fmpr_t *x*, double *v*)
Sets *x* the the exact value of the argument *v* of type *double*.

void **fmpr_set_ui** (fmpr_t *x*, ulong *c*)

void **fmpr_set_si** (fmpr_t *x*, long *c*)

void **fmpr_set_fmpz** (fmpr_t *x*, const fmpz_t *c*)
Sets *x* exactly to the integer *c*.

void **fmpr_get_fmpz** (fmpz_t *z*, const fmpr_t *x*, fmpr_rnd_t *rnd*)
Sets *z* to *x* rounded to the nearest integer in the direction specified by *rnd*. If *rnd* is *FMPR_RND_NEAR*, rounds to the nearest even integer in case of a tie. Aborts if *x* is infinite, NaN or if the exponent is unreasonably large.

long **fmpr_get_si** (const fmpr_t *x*, fmpr_rnd_t *rnd*)
Returns *x* rounded to the nearest integer in the direction specified by *rnd*. If *rnd* is *FMPR_RND_NEAR*, rounds to the nearest even integer in case of a tie. Aborts if *x* is infinite, NaN, or the value is too large to fit in a *long*.

void **fmpr_get_fmpq** (fmpq_t *y*, const fmpr_t *x*)
Sets *y* to the exact value of *x*. The result is undefined if *x* is not a finite fraction.

long **fmpr_set_fmpq** (fmpr_t *x*, const fmpq_t *y*, long *prec*, fmpr_rnd_t *rnd*)
Sets *x* to the value of *y*, rounded according to *prec* and *rnd*.

void **fmpr_set_fmpz_2exp** (fmpr_t *x*, const fmpz_t *man*, const fmpz_t *exp*)

void **fmpr_set_si_2exp_si** (fmpr_t *x*, long *man*, long *exp*)

void **fmpr_set_ui_2exp_si** (fmpr_t *x*, ulong *man*, long *exp*)
Sets *x* to $\mathrm{man} \times 2^{\mathrm{exp}}$.

long **fmpr_set_round_fmpz_2exp** (fmpr_t *x*, const fmpz_t *man*, const fmpz_t *exp*, long *prec*, fmpr_rnd_t *rnd*)
Sets *x* to $\mathrm{man} \times 2^{\mathrm{exp}}$, rounded according to *prec* and *rnd*.

void **fmpr_get_fmpz_2exp** (fmpz_t *man*, fmpz_t *exp*, const fmpr_t *x*)
Sets *man* and *exp* to the unique integers such that $x = \mathrm{man} \times 2^{\mathrm{exp}}$ and *man* is odd, provided that *x* is a nonzero finite fraction. If *x* is zero, both *man* and *exp* are set to zero. If *x* is infinite or NaN, the result is undefined.

int **fmpr_get_fmpz_fixed_fmpz** (fmpz_t *y*, const fmpr_t *x*, const fmpz_t *e*)

int **fmpr_get_fmpz_fixed_si** (fmpz_t *y*, const fmpr_t *x*, long *e*)
Converts *x* to a mantissa with predetermined exponent, i.e. computes an integer *y* such that $y \times 2^e \approx x$, truncating if necessary. Returns 0 if exact and 1 if truncation occurred.

5.1.5 Comparisons

int **fmpr_equal** (const fmpr_t *x*, const fmpr_t *y*)
Returns nonzero iff *x* and *y* are exactly equal. This function does not treat NaN specially, i.e. NaN compares as equal to itself.

int **fmpr_cmp** (const fmpr_t *x*, const fmpr_t *y*)
Returns negative, zero, or positive, depending on whether *x* is respectively smaller, equal, or greater compared to *y*. Comparison with NaN is undefined.

int **fmpr_cmpabs** (const fmpr_t *x*, const fmpr_t *y*)

int **fmpr_cmpabs_ui** (const fmpr_t *x*, ulong *y*)
Compares the absolute values of *x* and *y*.

int **fmpr_cmp_2exp_si** (const fmpr_t *x*, long *e*)

int **fmpr_cmpabs_2exp_si** (const fmpr_t *x*, long *e*)
Compares *x* (respectively its absolute value) with 2^e.

int **fmpr_sgn** (const fmpr_t *x*)
Returns -1, 0 or $+1$ according to the sign of *x*. The sign of NaN is undefined.

void **fmpr_min** (fmpr_t *z*, const fmpr_t *a*, const fmpr_t *b*)

void **fmpr_max** (fmpr_t *z*, const fmpr_t *a*, const fmpr_t *b*)
Sets *z* respectively to the minimum and the maximum of *a* and *b*.

long **fmpr_bits** (const fmpr_t *x*)
Returns the number of bits needed to represent the absolute value of the mantissa of *x*, i.e. the minimum precision sufficient to represent *x* exactly. Returns 0 if *x* is a special value.

int **fmpr_is_int** (const fmpr_t *x*)
Returns nonzero iff *x* is integer-valued.

int **fmpr_is_int_2exp_si** (const fmpr_t *x*, long *e*)
Returns nonzero iff *x* equals $n2^e$ for some integer *n*.

void **fmpr_abs_bound_le_2exp_fmpz** (fmpz_t *b*, const fmpr_t *x*)
Sets *b* to the smallest integer such that $|x| \leq 2^b$. If *x* is zero, infinity or NaN, the result is undefined.

void **fmpr_abs_bound_lt_2exp_fmpz** (fmpz_t *b*, const fmpr_t *x*)
Sets *b* to the smallest integer such that $|x| < 2^b$. If *x* is zero, infinity or NaN, the result is undefined.

long **fmpr_abs_bound_lt_2exp_si** (const fmpr_t *x*)
Returns the smallest integer *b* such that $|x| < 2^b$, clamping the result to lie between *-FMPR_PREC_EXACT* and *FMPR_PREC_EXACT* inclusive. If *x* is zero, *-FMPR_PREC_EXACT* is returned, and if *x* is infinity or NaN, *FMPR_PREC_EXACT* is returned.

5.1.6 Random number generation

void **fmpr_randtest** (fmpr_t *x*, flint_rand_t *state*, long *bits*, long *mag_bits*)
Generates a finite random number whose mantissa has precision at most *bits* and whose exponent has at most *mag_bits* bits. The values are distributed non-uniformly: special bit patterns are generated with high probability in order to allow the test code to exercise corner cases.

void **fmpr_randtest_not_zero** (fmpr_t *x*, flint_rand_t *state*, long *bits*, long *mag_bits*)
Identical to *fmpr_randtest*, except that zero is never produced as an output.

void **fmpr_randtest_special** (fmpr_t *x*, flint_rand_t *state*, long *bits*, long *mag_bits*)
Indentical to *fmpr_randtest*, except that the output occasionally is set to an infinity or NaN.

5.1.7 Input and output

void **fmpr_print** (const fmpr_t *x*)
Prints the mantissa and exponent of *x* as integers, precisely showing the internal representation.

void **fmpr_printd** (const fmpr_t *x*, long *digits*)
Prints *x* as a decimal floating-point number, rounding to the specified number of digits. This function is currently implemented using MPFR, and does not support large exponents.

5.1.8 Arithmetic

void **fmpr_neg** (fmpr_t *y*, const fmpr_t *x*)
Sets *y* to the negation of *x*.

long **fmpr_neg_round** (fmpr_t *y*, const fmpr_t *x*, long *prec*, fmpr_rnd_t *rnd*)
Sets *y* to the negation of *x*, rounding the result.

void **fmpr_abs** (fmpr_t *y*, const fmpr_t *x*)
Sets *y* to the absolute value of *x*.

long **fmpr_add** (fmpr_t *z*, const fmpr_t *x*, const fmpr_t *y*, long *prec*, fmpr_rnd_t *rnd*)

long **fmpr_add_ui** (fmpr_t *z*, const fmpr_t *x*, ulong *y*, long *prec*, fmpr_rnd_t *rnd*)

long **fmpr_add_si** (fmpr_t *z*, const fmpr_t *x*, long *y*, long *prec*, fmpr_rnd_t *rnd*)

long **fmpr_add_fmpz** (fmpr_t *z*, const fmpr_t *x*, const fmpz_t *y*, long *prec*, fmpr_rnd_t *rnd*)
Sets $z = x + y$, rounded according to *prec* and *rnd*. The precision can be *FMPR_PREC_EXACT* to perform an exact addition, provided that the result fits in memory.

long **_fmpr_add_eps** (fmpr_t *z*, const fmpr_t *x*, int *sign*, long *prec*, fmpr_rnd_t *rnd*)
Sets *z* to the value that results by adding an infinitesimal quantity of the given sign to *x*, and rounding. The result is undefined if *x* is zero.

long **fmpr_sub** (fmpr_t *z*, const fmpr_t *x*, const fmpr_t *y*, long *prec*, fmpr_rnd_t *rnd*)

long **fmpr_sub_ui** (fmpr_t *z*, const fmpr_t *x*, ulong *y*, long *prec*, fmpr_rnd_t *rnd*)

long **fmpr_sub_si** (fmpr_t *z*, const fmpr_t *x*, long *y*, long *prec*, fmpr_rnd_t *rnd*)

long **fmpr_sub_fmpz** (fmpr_t *z*, const fmpr_t *x*, const fmpz_t *y*, long *prec*, fmpr_rnd_t *rnd*)
Sets $z = x - y$, rounded according to *prec* and *rnd*. The precision can be *FMPR_PREC_EXACT* to perform an exact addition, provided that the result fits in memory.

long **fmpr_sum** (fmpr_t *s*, const fmpr_struct * *terms*, long *len*, long *prec*, fmpr_rnd_t *rnd*)
Sets *s* to the sum of the array *terms* of length *len*, rounded to *prec* bits in the direction *rnd*. The sum is computed as if done without any intermediate rounding error, with only a single rounding applied to the final result. Unlike repeated calls to *fmpr_add*, this function does not overflow if the magnitudes of the terms are far apart. Warning: this function is implemented naively, and the running time is quadratic with respect to *len* in the worst case.

long **fmpr_mul** (fmpr_t *z*, const fmpr_t *x*, const fmpr_t *y*, long *prec*, fmpr_rnd_t *rnd*)

long **fmpr_mul_ui** (fmpr_t *z*, const fmpr_t *x*, ulong *y*, long *prec*, fmpr_rnd_t *rnd*)

long **fmpr_mul_si** (fmpr_t *z*, const fmpr_t *x*, long *y*, long *prec*, fmpr_rnd_t *rnd*)

long **fmpr_mul_fmpz** (fmpr_t *z*, const fmpr_t *x*, const fmpz_t *y*, long *prec*, fmpr_rnd_t *rnd*)
Sets $z = x \times y$, rounded according to prec and rnd. The precision can be *FMPR_PREC_EXACT* to perform an exact multiplication, provided that the result fits in memory.

void **fmpr_mul_2exp_si** (fmpr_t *y*, const fmpr_t *x*, long *e*)

void **fmpr_mul_2exp_fmpz** (fmpr_t *y*, const fmpr_t *x*, const fmpz_t *e*)
Sets *y* to *x* multiplied by 2^e without rounding.

long **fmpr_div** (fmpr_t *z*, const fmpr_t *x*, const fmpr_t *y*, long *prec*, fmpr_rnd_t *rnd*)

long **fmpr_div_ui** (fmpr_t *z*, const fmpr_t *x*, ulong *y*, long *prec*, fmpr_rnd_t *rnd*)

long **fmpr_ui_div** (fmpr_t *z*, ulong *x*, const fmpr_t *y*, long *prec*, fmpr_rnd_t *rnd*)

long **fmpr_div_si** (fmpr_t *z*, const fmpr_t *x*, long *y*, long *prec*, fmpr_rnd_t *rnd*)

long **fmpr_si_div** (fmpr_t *z*, long *x*, const fmpr_t *y*, long *prec*, fmpr_rnd_t *rnd*)

long **fmpr_div_fmpz** (fmpr_t *z*, const fmpr_t *x*, const fmpz_t *y*, long *prec*, fmpr_rnd_t *rnd*)

long **fmpr_fmpz_div** (fmpr_t *z*, const fmpz_t *x*, const fmpr_t *y*, long *prec*, fmpr_rnd_t *rnd*)

long **fmpr_fmpz_div_fmpz** (fmpr_t *z*, const fmpz_t *x*, const fmpz_t *y*, long *prec*, fmpr_rnd_t *rnd*)
Sets $z = x/y$, rounded according to *prec* and *rnd*. If *y* is zero, *z* is set to NaN.

void **fmpr_divappr_abs_ubound** (fmpr_t *z*, const fmpr_t *x*, const fmpr_t *y*, long *prec*)
Sets z to an upper bound for $|x|/|y|$, computed to a precision of approximately *prec* bits. The error can be a few ulp.

long **fmpr_addmul** (fmpr_t *z*, const fmpr_t *x*, const fmpr_t *y*, long *prec*, fmpr_rnd_t *rnd*)

long **fmpr_addmul_ui** (fmpr_t *z*, const fmpr_t *x*, ulong *y*, long *prec*, fmpr_rnd_t *rnd*)

long **fmpr_addmul_si** (fmpr_t *z*, const fmpr_t *x*, long *y*, long *prec*, fmpr_rnd_t *rnd*)

long **fmpr_addmul_fmpz** (fmpr_t *z*, const fmpr_t *x*, const fmpz_t *y*, long *prec*, fmpr_rnd_t *rnd*)
Sets $z = z + x \times y$, rounded according to *prec* and *rnd*. The intermediate multiplication is always performed without roundoff. The precision can be *FMPR_PREC_EXACT* to perform an exact addition, provided that the result fits in memory.

long **fmpr_submul** (fmpr_t *z*, const fmpr_t *x*, const fmpr_t *y*, long *prec*, fmpr_rnd_t *rnd*)

long **fmpr_submul_ui** (fmpr_t *z*, const fmpr_t *x*, ulong *y*, long *prec*, fmpr_rnd_t *rnd*)

long **fmpr_submul_si** (fmpr_t *z*, const fmpr_t *x*, long *y*, long *prec*, fmpr_rnd_t *rnd*)

long **fmpr_submul_fmpz** (fmpr_t *z*, const fmpr_t *x*, const fmpz_t *y*, long *prec*, fmpr_rnd_t *rnd*)
Sets $z = z - x \times y$, rounded according to *prec* and *rnd*. The intermediate multiplication is always performed without roundoff. The precision can be *FMPR_PREC_EXACT* to perform an exact subtraction, provided that the result fits in memory.

long **fmpr_sqrt** (fmpr_t *y*, const fmpr_t *x*, long *prec*, fmpr_rnd_t *rnd*)

long **fmpr_sqrt_ui** (fmpr_t *z*, ulong *x*, long *prec*, fmpr_rnd_t *rnd*)

long **fmpr_sqrt_fmpz** (fmpr_t *z*, const fmpz_t *x*, long *prec*, fmpr_rnd_t *rnd*)
Sets *z* to the square root of *x*, rounded according to *prec* and *rnd*. The result is NaN if *x* is negative.

long **fmpr_rsqrt** (fmpr_t *z*, const fmpr_t *x*, long *prec*, fmpr_rnd_t *rnd*)
Sets *z* to the reciprocal square root of *x*, rounded according to *prec* and *rnd*. The result is NaN if *x* is negative. At high precision, this is faster than computing a square root.

long **fmpr_root** (fmpr_t *z*, const fmpr_t *x*, ulong *k*, long *prec*, fmpr_rnd_t *rnd*)
Sets *z* to the *k*-th root of *x*, rounded to *prec* bits in the direction *rnd*. Warning: this function wraps MPFR, and is currently only fast for small *k*.

void **fmpr_pow_sloppy_fmpz** (fmpr_t *y*, const fmpr_t *b*, const fmpz_t *e*, long *prec*, fmpr_rnd_t *rnd*)

void **fmpr_pow_sloppy_ui** (fmpr_t *y*, const fmpr_t *b*, ulong *e*, long *prec*, fmpr_rnd_t *rnd*)

void **fmpr_pow_sloppy_si** (fmpr_t *y*, const fmpr_t *b*, long *e*, long *prec*, fmpr_rnd_t *rnd*)
Sets $y = b^e$, computed using without guaranteeing correct (optimal) rounding, but guaranteeing that the result is a correct upper or lower bound if the rounding is directional. Currently requires $b \geq 0$.

5.1.9 Special functions

long **fmpr_log** (fmpr_t *y*, const fmpr_t *x*, long *prec*, fmpr_rnd_t *rnd*)
Sets *y* to $\log(x)$, rounded according to *prec* and *rnd*. The result is NaN if *x* is negative. This function is currently implemented using MPFR and does not support large exponents.

long **fmpr_log1p** (fmpr_t *y*, const fmpr_t *x*, long *prec*, fmpr_rnd_t *rnd*)
Sets *y* to $\log(1 + x)$, rounded according to *prec* and *rnd*. This function computes an accurate value when *x* is small. The result is NaN if $1 + x$ is negative. This function is currently implemented using MPFR and does not support large exponents.

long **fmpr_exp** (fmpr_t *y*, const fmpr_t *x*, long *prec*, fmpr_rnd_t *rnd*)
Sets *y* to $\exp(x)$, rounded according to *prec* and *rnd*. This function is currently implemented using MPFR and does not support large exponents.

long **fmpr_expm1** (fmpr_t *y*, const fmpr_t *x*, long *prec*, fmpr_rnd_t *rnd*)
Sets *y* to $\exp(x) - 1$, rounded according to *prec* and *rnd*. This function computes an accurate value when *x* is small. This function is currently implemented using MPFR and does not support large exponents.

5.2 fmprb.h – real numbers represented as floating-point balls

This type is now obsolete: use `arb_t` instead.

An `fmprb_t` represents a ball over the real numbers.

5.2.1 Types, macros and constants

fmprb_struct

fmprb_t
An *fmprb_struct* consists of a pair of *fmpr_struct*:s. An *fmprb_t* is defined as an array of length one of type *fmprb_struct*, permitting an *fmprb_t* to be passed by reference.

fmprb_ptr
Alias for `fmprb_struct *`, used for vectors of numbers.

fmprb_srcptr
Alias for `const fmprb_struct *`, used for vectors of numbers when passed as constant input to functions.

FMPRB_RAD_PREC
The precision used for operations on the radius. This is small enough to fit in a single word, currently 30 bits.

fmprb_midref (x)
Macro returning a pointer to the midpoint of *x* as an *fmpr_t*.

fmprb_radref (x)
Macro returning a pointer to the radius of *x* as an *fmpr_t*.

5.2.2 Memory management

void **fmprb_init** (fmprb_t *x*)
Initializes the variable *x* for use. Its midpoint and radius are both set to zero.

void **fmprb_clear** (fmprb_t *x*)
Clears the variable *x*, freeing or recycling its allocated memory.

fmprb_ptr **_fmprb_vec_init** (long *n*)
Returns a pointer to an array of *n* initialized *fmprb_struct*:s.

void **_fmprb_vec_clear** (fmprb_ptr *v*, long *n*)
Clears an array of *n* initialized *fmprb_struct*:s.

5.2.3 Assignment and rounding

void **fmprb_set** (fmprb_t *y*, const fmprb_t *x*)
Sets *y* to a copy of *x*.

void **fmprb_set_round** (fmprb_t *y*, const fmprb_t *x*, long *prec*)
Sets *y* to a copy of *x*, rounded to *prec* bits.

void **fmprb_set_fmpr** (fmprb_t *y*, const fmpr_t *x*)

void **fmprb_set_si** (fmprb_t *y*, long *x*)

void **fmprb_set_ui** (fmprb_t *y*, ulong *x*)

void **fmprb_set_fmpz** (fmprb_t *y*, const fmpz_t *x*)
Sets *y* exactly to *x*.

void **fmprb_set_fmpq** (fmprb_t *y*, const fmpq_t *x*, long *prec*)
Sets *y* to the rational number *x*, rounded to *prec* bits.

void **fmprb_set_fmpz_2exp** (fmprb_t *x*, const fmpz_t *y*, const fmpz_t *exp*)
Sets *x* to *y* multiplied by 2 raised to the power *exp*.

void **fmprb_set_round_fmpz_2exp** (fmprb_t *y*, const fmpz_t *x*, const fmpz_t *exp*, long *prec*)
Sets *x* to *y* multiplied by 2 raised to the power *exp*, rounding the result to *prec* bits.

5.2.4 Assignment of special values

void **fmprb_zero** (fmprb_t *x*)
Sets *x* to zero.

void **fmprb_one** (fmprb_t *x*)
Sets *x* to the exact integer 1.

void **fmprb_pos_inf** (fmprb_t *x*)
Sets *x* to positive infinity, with a zero radius.

void **fmprb_neg_inf** (fmprb_t *x*)
Sets *x* to negative infinity, with a zero radius.

void **fmprb_zero_pm_inf** (fmprb_t *x*)
Sets *x* to $[0 \pm \infty]$, representing the whole extended real line.

void **fmprb_indeterminate** (fmprb_t *x*)
Sets *x* to $[\mathrm{NaN} \pm \infty]$, representing an indeterminate result.

5.2.5 Input and output

void **fmprb_print** (const fmprb_t *x*)
Prints the internal representation of *x*.

void **fmprb_printd** (const fmprb_t *x*, long *digits*)
Prints *x* in decimal. The printed value of the radius is not adjusted to compensate for the fact that the binary-to-decimal conversion of both the midpoint and the radius introduces additional error.

5.2.6 Random number generation

void **fmprb_randtest** (fmprb_t *x*, flint_rand_t *state*, long *prec*, long *mag_bits*)
Generates a random ball. The midpoint and radius will both be finite.

void **fmprb_randtest_exact** (fmprb_t *x*, flint_rand_t *state*, long *prec*, long *mag_bits*)
Generates a random number with zero radius.

void **fmprb_randtest_precise** (fmprb_t *x*, flint_rand_t *state*, long *prec*, long *mag_bits*)
Generates a random number with radius at most $2^{-\mathrm{prec}}$ the magnitude of the midpoint.

void **fmprb_randtest_wide** (fmprb_t *x*, flint_rand_t *state*, long *prec*, long *mag_bits*)
Generates a random number with midpoint and radius chosen independently, possibly giving a very large interval.

void **fmprb_randtest_special** (fmprb_t *x*, flint_rand_t *state*, long *prec*, long *mag_bits*)
Generates a random interval, possibly having NaN or an infinity as the midpoint and possibly having an infinite radius.

void **fmprb_get_rand_fmpq** (fmpq_t *q*, flint_rand_t *state*, const fmprb_t *x*, long *bits*)
Sets *q* to a random rational number from the interval represented by *x*. A denominator is chosen by multiplying the binary denominator of *x* by a random integer up to *bits* bits.

The outcome is undefined if the midpoint or radius of *x* is non-finite, or if the exponent of the midpoint or radius is so large or small that representing the endpoints as exact rational numbers would cause overflows.

5.2.7 Radius and interval operations

void **fmprb_add_error_fmpr** (fmprb_t *x*, const fmpr_t *err*)
Adds *err*, which is assumed to be nonnegative, to the radius of *x*.

void **fmprb_add_error_2exp_si** (fmprb_t *x*, long *e*)

void **fmprb_add_error_2exp_fmpz** (fmprb_t *x*, const fmpz_t *e*)
Adds 2^e to the radius of *x*.

void **fmprb_add_error** (fmprb_t *x*, const fmprb_t *err*)
Adds the supremum of *err*, which is assumed to be nonnegative, to the radius of *x*.

void **fmprb_union** (fmprb_t *z*, const fmprb_t *x*, const fmprb_t *y*, long *prec*)
Sets *z* to a ball containing both *x* and *y*.

void **fmprb_get_abs_ubound_fmpr** (fmpr_t *u*, const fmprb_t *x*, long *prec*)
Sets *u* to the upper bound of the absolute value of *x*, rounded up to *prec* bits. If *x* contains NaN, the result is NaN.

void **fmprb_get_abs_lbound_fmpr** (fmpr_t *u*, const fmprb_t *x*, long *prec*)
Sets *u* to the lower bound of the absolute value of *x*, rounded down to *prec* bits. If *x* contains NaN, the result is NaN.

void **fmprb_get_interval_fmpz_2exp** (fmpz_t *a*, fmpz_t *b*, fmpz_t *exp*, const fmprb_t *x*)
Computes the exact interval represented by *x*, in the form of an integer interval multiplied by a power of two, i.e. $x = [a, b] \times 2^{\exp}$.

The outcome is undefined if the midpoint or radius of *x* is non-finite, or if the difference in magnitude between the midpoint and radius is so large that representing the endpoints exactly would cause overflows.

void **fmprb_set_interval_fmpr** (fmprb_t *x*, const fmpr_t *a*, const fmpr_t *b*, long *prec*)
Sets *x* to a ball containing the interval $[a, b]$. We require that $a \leq b$.

long **fmprb_rel_error_bits** (const fmprb_t *x*)
Returns the effective relative error of *x* measured in bits, defined as the difference between the position of the top bit in the radius and the top bit in the midpoint, plus one. The result is clamped between plus/minus *FMPR_PREC_EXACT*.

long **fmprb_rel_accuracy_bits** (const fmprb_t *x*)
Returns the effective relative accuracy of *x* measured in bits, equal to the negative of the return value from *fmprb_rel_error_bits*.

long **fmprb_bits** (const fmprb_t *x*)
Returns the number of bits needed to represent the absolute value of the mantissa of the midpoint of *x*, i.e. the minimum precision sufficient to represent *x* exactly. Returns 0 if the midpoint of *x* is a special value.

void **fmprb_trim** (fmprb_t *y*, const fmprb_t *x*)
Sets *y* to a trimmed copy of *x*: rounds *x* to a number of bits equal to the accuracy of *x* (as indicated by its radius), plus a few guard bits. The resulting ball is guaranteed to contain *x*, but is more economical if *x* has less than full accuracy.

int **fmprb_get_unique_fmpz** (fmpz_t *z*, const fmprb_t *x*)
If *x* contains a unique integer, sets *z* to that value and returns nonzero. Otherwise (if *x* represents no integers or more than one integer), returns zero.

5.2.8 Comparisons

int **fmprb_is_zero** (const fmprb_t *x*)
Returns nonzero iff the midpoint and radius of *x* are both zero.

int **fmprb_is_nonzero** (const fmprb_t *x*)
Returns nonzero iff zero is not contained in the interval represented by *x*.

int **fmprb_is_one** (const fmprb_t *x*)
Returns nonzero iff *x* is exactly 1.

int **fmprb_is_finite** (fmprb_t *x*)
Returns nonzero iff the midpoint and radius of *x* are both finite floating-point numbers, i.e. not infinities or NaN.

int **fmprb_is_exact** (const fmprb_t *x*)
Returns nonzero iff the radius of *x* is zero.

int **fmprb_is_int** (const fmprb_t *x*)
Returns nonzero iff *x* is an exact integer.

int **fmprb_equal** (const fmprb_t *x*, const fmprb_t *y*)
Returns nonzero iff *x* and *y* are equal as balls, i.e. have both the same midpoint and radius.

Note that this is not the same thing as testing whether both *x* and *y* certainly represent the same real number, unless either *x* or *y* is exact (and neither contains NaN). To test whether both operands *might* represent the same mathematical quantity, use `fmprb_overlaps()` or `fmprb_contains()`, depending on the circumstance.

int **fmprb_is_positive** (const fmprb_t *x*) int **fmprb_is_nonnegative** (const fmprb_t *x*)

int **fmprb_is_negative** (const fmprb_t *x*)

int **fmprb_is_nonpositive** (const fmprb_t *x*)
Returns nonzero iff all points *p* in the interval represented by *x* satisfy, respectively, $p > 0$, $p \geq 0$, $p < 0$, $p \leq 0$. If *x* contains NaN, returns zero.

int **fmprb_overlaps** (const fmprb_t *x*, const fmprb_t *y*)
Returns nonzero iff *x* and *y* have some point in common. If either *x* or *y* contains NaN, this function always returns nonzero (as a NaN could be anything, it could in particular contain any number that is included in the other operand).

int **fmprb_contains_fmpr** (const fmprb_t *x*, const fmpr_t *y*)

int **fmprb_contains_fmpq** (const fmprb_t *x*, const fmpq_t *y*)

int **fmprb_contains_fmpz** (const fmprb_t *x*, const fmpz_t *y*)

int **fmprb_contains_si** (const fmprb_t *x*, long *y*)

int **fmprb_contains_mpfr** (const fmprb_t *x*, const mpfr_t *y*)

int **fmprb_contains_zero** (const fmprb_t *x*)

int **fmprb_contains** (const fmprb_t *x*, const fmprb_t *y*)
Returns nonzero iff the given number (or ball) *y* is contained in the interval represented by *x*.

If *x* is contains NaN, this function always returns nonzero (as it could represent anything, and in particular could represent all the points included in *y*). If *y* contains NaN and *x* does not, it always returns zero.

int **fmprb_contains_negative** (const fmprb_t *x*)

int **fmprb_contains_nonpositive** (const fmprb_t *x*)

int **fmprb_contains_positive** (const fmprb_t *x*)

int **fmprb_contains_nonnegative** (const fmprb_t *x*)
Returns nonzero iff there is any point *p* in the interval represented by *x* satisfying, respectively, $p < 0$, $p \leq 0$, $p > 0$, $p \geq 0$. If *x* contains NaN, returns nonzero.

5.2.9 Arithmetic

void **fmprb_neg** (fmprb_t *y*, const fmprb_t *x*)
Sets *y* to the negation of *x*.

void **fmprb_abs** (fmprb_t *y*, const fmprb_t *x*)
Sets *y* to the absolute value of *x*. No attempt is made to improve the interval represented by *x* if it contains zero.

void **fmprb_add** (fmprb_t *z*, const fmprb_t *x*, const fmprb_t *y*, long *prec*)

void **fmprb_add_ui** (fmprb_t *z*, const fmprb_t *x*, ulong *y*, long *prec*)

void **fmprb_add_si** (fmprb_t *z*, const fmprb_t *x*, long *y*, long *prec*)

void **fmprb_add_fmpz** (fmprb_t *z*, const fmprb_t *x*, const fmpz_t *y*, long *prec*)

void **fmprb_add_fmpr** (fmprb_t *z*, const fmprb_t *x*, const fmpr_t *y*, long *prec*)
Sets $z = x + y$, rounded to *prec* bits. The precision can be *FMPR_PREC_EXACT* provided that the result fits in memory.

void **fmprb_sub** (fmprb_t *z*, const fmprb_t *x*, const fmprb_t *y*, long *prec*)

void **fmprb_sub_ui** (fmprb_t *z*, const fmprb_t *x*, ulong *y*, long *prec*) void **fmprb_sub_si** (fmprb_t *z*, const fmprb_t *x*, long *y*, long *prec*)

void **fmprb_sub_fmpz** (fmprb_t *z*, const fmprb_t *x*, const fmpz_t *y*, long *prec*)
Sets $z = x - y$, rounded to *prec* bits. The precision can be *FMPR_PREC_EXACT* provided that the result fits in memory.

void **fmprb_mul** (fmprb_t *z*, const fmprb_t *x*, const fmprb_t *y*, long *prec*)

void **fmprb_mul_ui** (fmprb_t *z*, const fmprb_t *x*, ulong *y*, long *prec*)

void **fmprb_mul_si** (fmprb_t *z*, const fmprb_t *x*, long *y*, long *prec*)

void **fmprb_mul_fmpz** (fmprb_t *z*, const fmprb_t *x*, const fmpz_t *y*, long *prec*)
Sets $z = x \times y$, rounded to *prec* bits. The precision can be *FMPR_PREC_EXACT* provided that the result fits in memory.

void **fmprb_mul_2exp_si** (fmprb_t *y*, const fmprb_t *x*, long *e*)

void **fmprb_mul_2exp_fmpz** (fmprb_t *y*, const fmprb_t *x*, const fmpz_t *e*)
Sets *y* to *x* multiplied by 2^e.

void **fmprb_inv** (fmprb_t *z*, const fmprb_t *x*, long *prec*)
Sets *z* to the multiplicative inverse of *x*.

void **fmprb_div** (fmprb_t *z*, const fmprb_t *x*, const fmprb_t *y*, long *prec*)

void **fmprb_div_ui** (fmprb_t *z*, const fmprb_t *x*, ulong *y*, long *prec*)

void **fmprb_div_si** (fmprb_t *z*, const fmprb_t *x*, long *y*, long *prec*)

void **fmprb_div_fmpz** (fmprb_t *z*, const fmprb_t *x*, const fmpz_t *y*, long *prec*)

void **fmprb_div_fmpr** (fmprb_t *z*, const fmprb_t *x*, const fmpr_t *y*, long *prec*)

void **fmprb_fmpz_div_fmpz** (fmprb_t *y*, const fmpz_t *num*, const fmpz_t *den*, long *prec*)

void **fmprb_ui_div** (fmprb_t *z*, ulong *x*, const fmprb_t *y*, long *prec*)
Sets $z = x/y$, rounded to *prec* bits. If *y* contains zero, *z* is set to $0 \pm \infty$. Otherwise, error propagation uses the rule

$$\left|\frac{x}{y} - \frac{x+\xi_1 a}{y+\xi_2 b}\right| = \left|\frac{x\xi_2 b - y\xi_1 a}{y(y+\xi_2 b)}\right| \le \frac{|xb|+|ya|}{|y|(|y|-b)}$$

where $-1 \le \xi_1, \xi_2 \le 1$, and where the triangle inequality has been applied to the numerator and the reverse triangle inequality has been applied to the denominator.

void **fmprb_div_2expm1_ui** (fmprb_t *y*, const fmprb_t *x*, ulong *n*, long *prec*)
Sets $y = x/(2^n - 1)$, rounded to *prec* bits.

void **fmprb_addmul** (fmprb_t *z*, const fmprb_t *x*, const fmprb_t *y*, long *prec*)

void **fmprb_addmul_ui** (fmprb_t *z*, const fmprb_t *x*, ulong *y*, long *prec*)

void **fmprb_addmul_si** (fmprb_t *z*, const fmprb_t *x*, long *y*, long *prec*)

void **fmprb_addmul_fmpz** (fmprb_t *z*, const fmprb_t *x*, const fmpz_t *y*, long *prec*)
Sets $z = z + x \times y$, rounded to prec bits. The precision can be *FMPR_PREC_EXACT* provided that the result fits in memory.

void **fmprb_submul** (fmprb_t *z*, const fmprb_t *x*, const fmprb_t *y*, long *prec*)

void **fmprb_submul_ui** (fmprb_t *z*, const fmprb_t *x*, ulong *y*, long *prec*)

void **fmprb_submul_si** (fmprb_t *z*, const fmprb_t *x*, long *y*, long *prec*)

void **fmprb_submul_fmpz** (fmprb_t *z*, const fmprb_t *x*, const fmpz_t *y*, long *prec*)
Sets $z = z - x \times y$, rounded to *prec* bits. The precision can be *FMPR_PREC_EXACT* provided that the result fits in memory.

5.2.10 Powers and roots

void **fmprb_sqrt** (fmprb_t *z*, const fmprb_t *x*, long *prec*)

void **fmprb_sqrt_ui** (fmprb_t *z*, ulong *x*, long *prec*)

void **fmprb_sqrt_fmpz** (fmprb_t *z*, const fmpz_t *x*, long *prec*)
Sets *z* to the square root of *x*, rounded to *prec* bits. Error propagation is done using the following rule: assuming $m > r \geq 0$, the error is largest at $m - r$, and we have $\sqrt{m} - \sqrt{m-r} \leq r/(2\sqrt{m-r})$.

void **fmprb_sqrtpos** (fmprb_t *z*, const fmprb_t *x*, long *prec*)
Sets *z* to the square root of *x*, assuming that *x* represents a nonnegative number (i.e. discarding any negative numbers in the input interval), and producing an output interval not containing any negative numbers (unless the radius is infinite).

void **fmprb_hypot** (fmprb_t *z*, const fmprb_t *x*, const fmprb_t *y*, long *prec*)
Sets *z* to $\sqrt{x^2 + y^2}$.

void **fmprb_rsqrt** (fmprb_t *z*, const fmprb_t *x*, long *prec*)

void **fmprb_rsqrt_ui** (fmprb_t *z*, ulong *x*, long *prec*)
Sets *z* to the reciprocal square root of *x*, rounded to *prec* bits. At high precision, this is faster than computing a square root.

void **fmprb_root** (fmprb_t *z*, const fmprb_t *x*, ulong *k*, long *prec*)
Sets *z* to the *k*-th root of *x*, rounded to *prec* bits. As currently implemented, this function is only fast for small fixed *k*. For large *k* it is better to use `fmprb_pow_fmpq()` or `fmprb_pow()`.

void **fmprb_agm** (fmprb_t *z*, const fmprb_t *x*, const fmprb_t *y*, long *prec*)
Sets *z* to the arithmetic-geometric mean of *x* and *y*.

CHAPTER

SIX

CREDITS AND REFERENCES

6.1 Credits and references

Arb is licensed GNU General Public License version 2, or any later version.

Arb includes code by Bill Hart and Sebastian Pancratz taken from FLINT (also licensed GPL 2.0+).

From 2012 to July 2014, Fredrik's work on Arb was supported by Austrian Science Fund FWF Grant Y464-N18 (Fast Computer Algebra for Special Functions). During that period, he was a PhD student (and briefly a postdoc) at RISC, Johannes Kepler University, Linz, supervised by Manuel Kauers.

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6.1.1 Contributors

The following people (among others) have contributed patches or bug reports.

- Jonathan Bober
- Yuri Matiyasevich
- Abhinav Baid
- Ondřej Čertík
- Andrew Booker
- Francesco Biscani
- Clemens Heuberger
- Pascal Molin
- Ricky Farr

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- MPFR (Guillaume Hanrot, Vincent Lefèvre, Patrick Pélissier, Philippe Théveny, Paul Zimmermann and others), http://mpfr.org

- FLINT (William Hart, Sebastian Pancratz, Andy Novocin, Fredrik Johansson, David Harvey and others), http://flintlib.org
- Sage (William Stein and others), http://sagemath.org
- Pari/GP (The Pari group), http://pari.math.u-bordeaux.fr/
- SymPy (Ondřej Čertík, Aaron Meurer and others), http://sympy.org
- mpmath (Fredrik Johansson and others), http://mpmath.org
- Mathematica (Wolfram Research), http://www.wolfram.com/mathematica
- HolonomicFunctions (Christoph Koutschan), http://www.risc.jku.at/research/combinat/software/HolonomicFunctions/
- Sphinx (George Brandl and others), http://sphinx.pocoo.org
- CM (Andreas Enge), http://www.multiprecision.org/index.php?prog=cm

6.1.3 Citing Arb

If you wish to cite Arb in a scientific paper, the following reference can be used (you may also cite the manual or the website directly):

F. Johansson. "Arb: a C library for ball arithmetic", *ACM Communications in Computer Algebra*, 47(4):166-169, 2013.

In BibTeX format:

```
@article{Johansson2013arb,
  title={{A}rb: a {C} library for ball arithmetic},
  author={F. Johansson},
  journal={ACM Communications in Computer Algebra},
  volume={47},
  number={4},
  pages={166--169},
  year={2013},
  publisher={ACM}
}
```

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Symbols

A

B

F

H

M

P